Cartoon Physics

Mechanics

Written and illustrated by

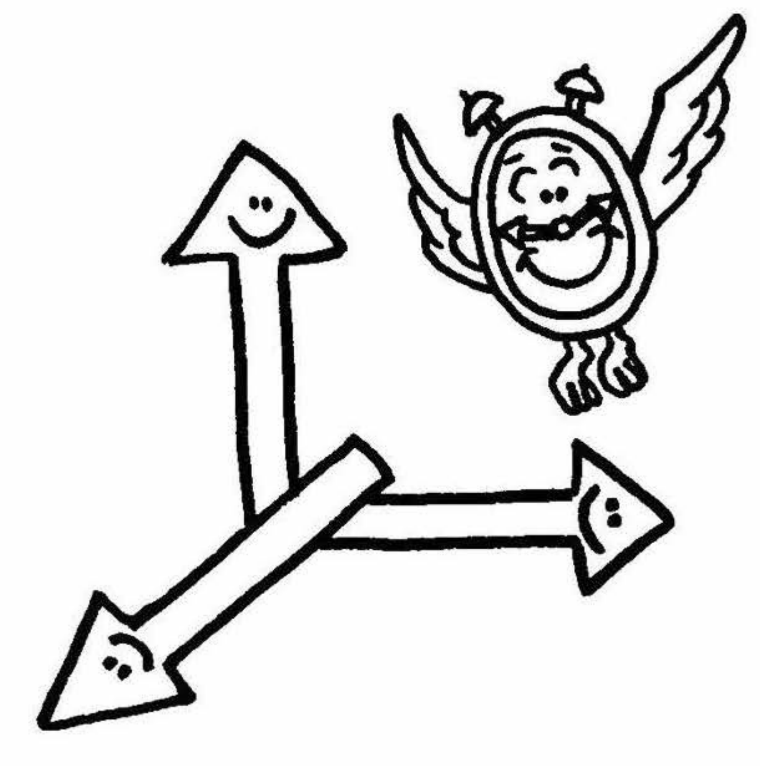
Salma Alrasheed, PhD

ISBN-13: 978-1-7986-5580-1

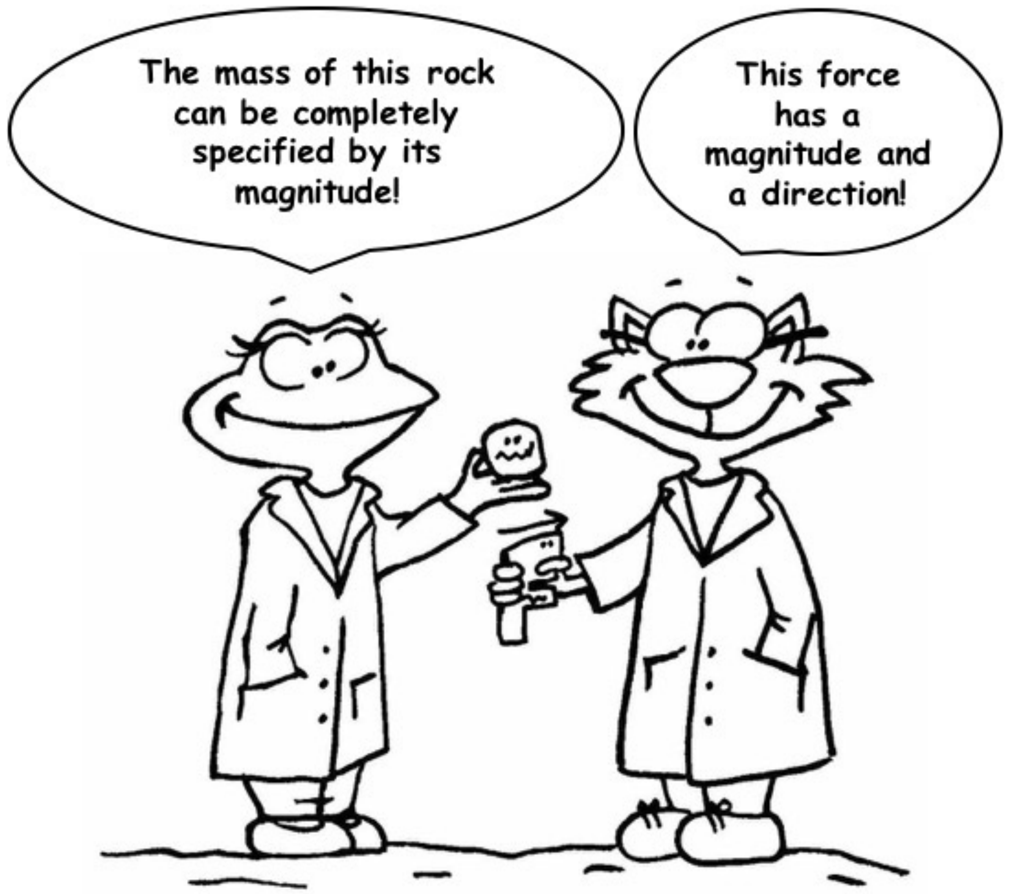
Contents

Kinematics

The first concept we want to discuss is how to describe a physical quantity. We live in a world that has at least four dimensions, three of space and one of time.

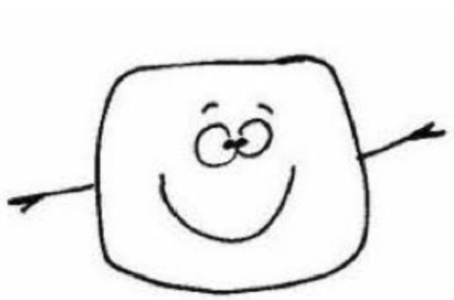


From observations, it is found that some quantities known as scalars need only a magnitude along with its units to be completely described, while others known as vectors need a magnitude with its units in addition to a specific direction. Scalars follow the rules of ordinary arithmetic and vectors follow the rules of vector algebra.



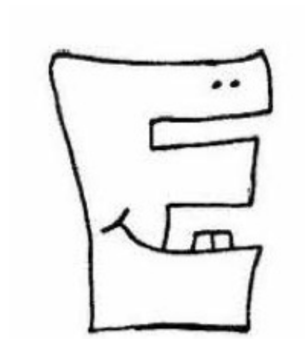
An arrow is placed over a letter representing a vector (such as $\vec{A}$) to distinguish it from a scalar. The magnitude of a vector $\vec{A}$ is written as A or $|\vec{A}|$.

Examples of Scalars:

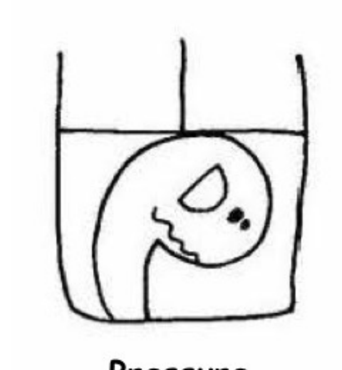


Volume, unit: m^3

Temperature, unit: Kelvin (K)

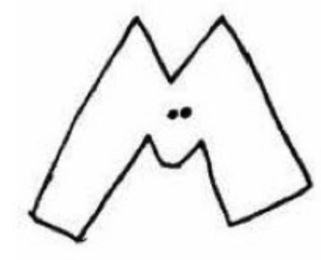


Energy, unit: Joule (J)



Pressure, unit: Pascal (Pa)

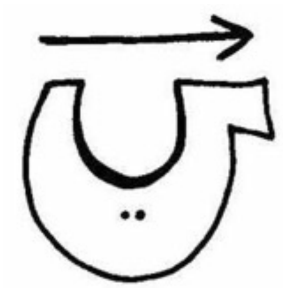
Time, unit: Second (s)



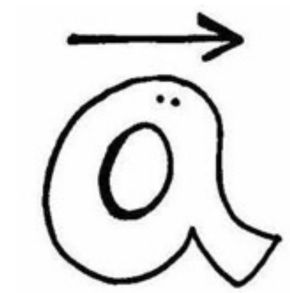
Mass, unit: Kilogram (kg)

Examples of Vectors:

Force, unit: Newton (N)



Velocity, unit: m/s



Acceleration, unit: m/s^2

Electric Field, unit: Newtons per Coulomb (N/C) or Volts per Meter (V/m)

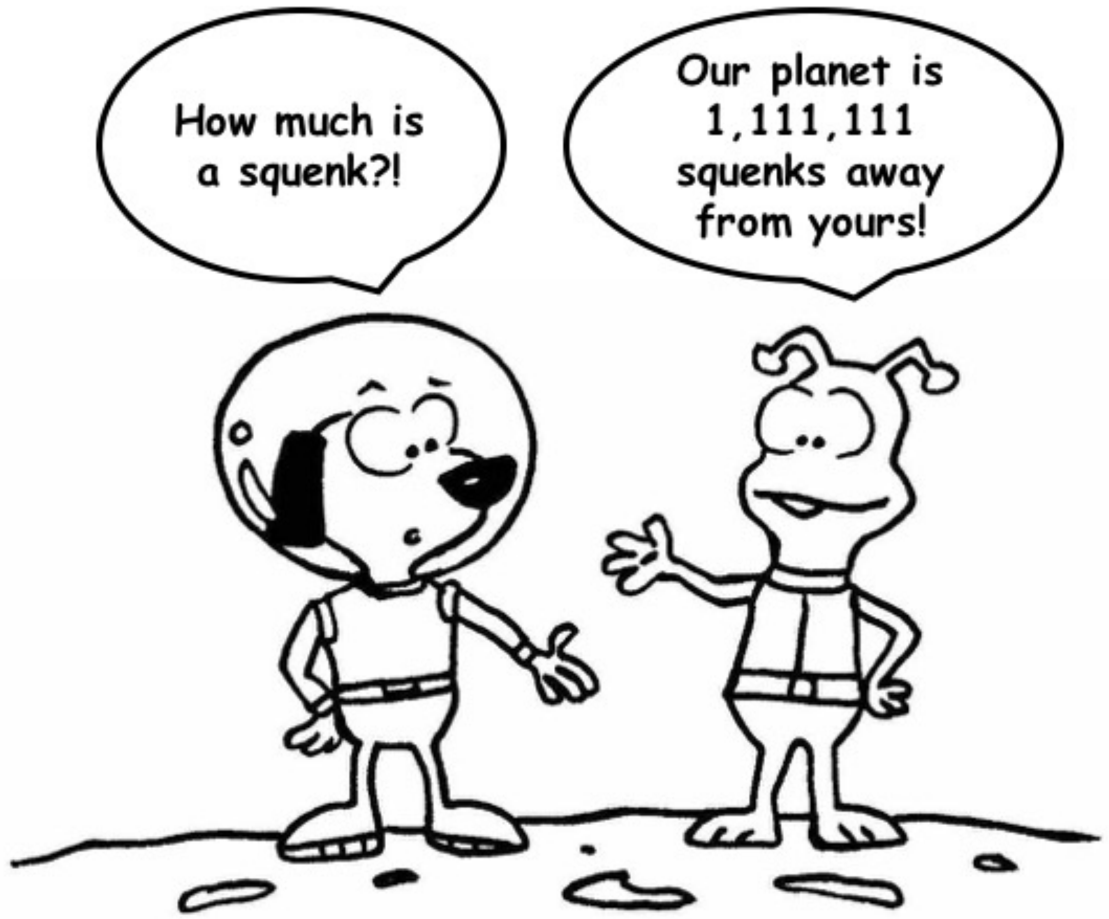
Magnetic Flux Density, unit: Tesla (T)

Momentum, unit: kg.m/s

Making a measurement: Measuring something means comparing the quantity with a standard called a unit.

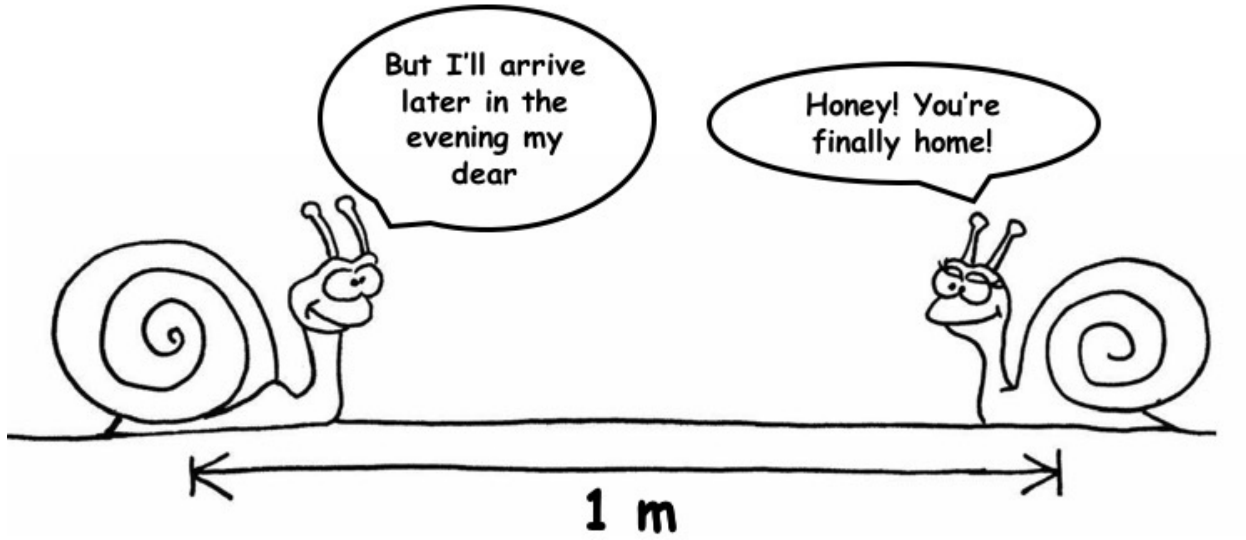
The Elephant weighs 5,400 kg, which is equivalent to 5,400 kg platinum-iridium alloy cylinders (see below for the definition of the kilogram unit).

It would be much easier for communication if people (or aliens or animals) would use the same units.



This is why the international system of units (abbreviated the SI system from the French phrase Système International) was established. In this system, the units used for length, mass and time are the meter, kilogram and second respectively defined as follows:

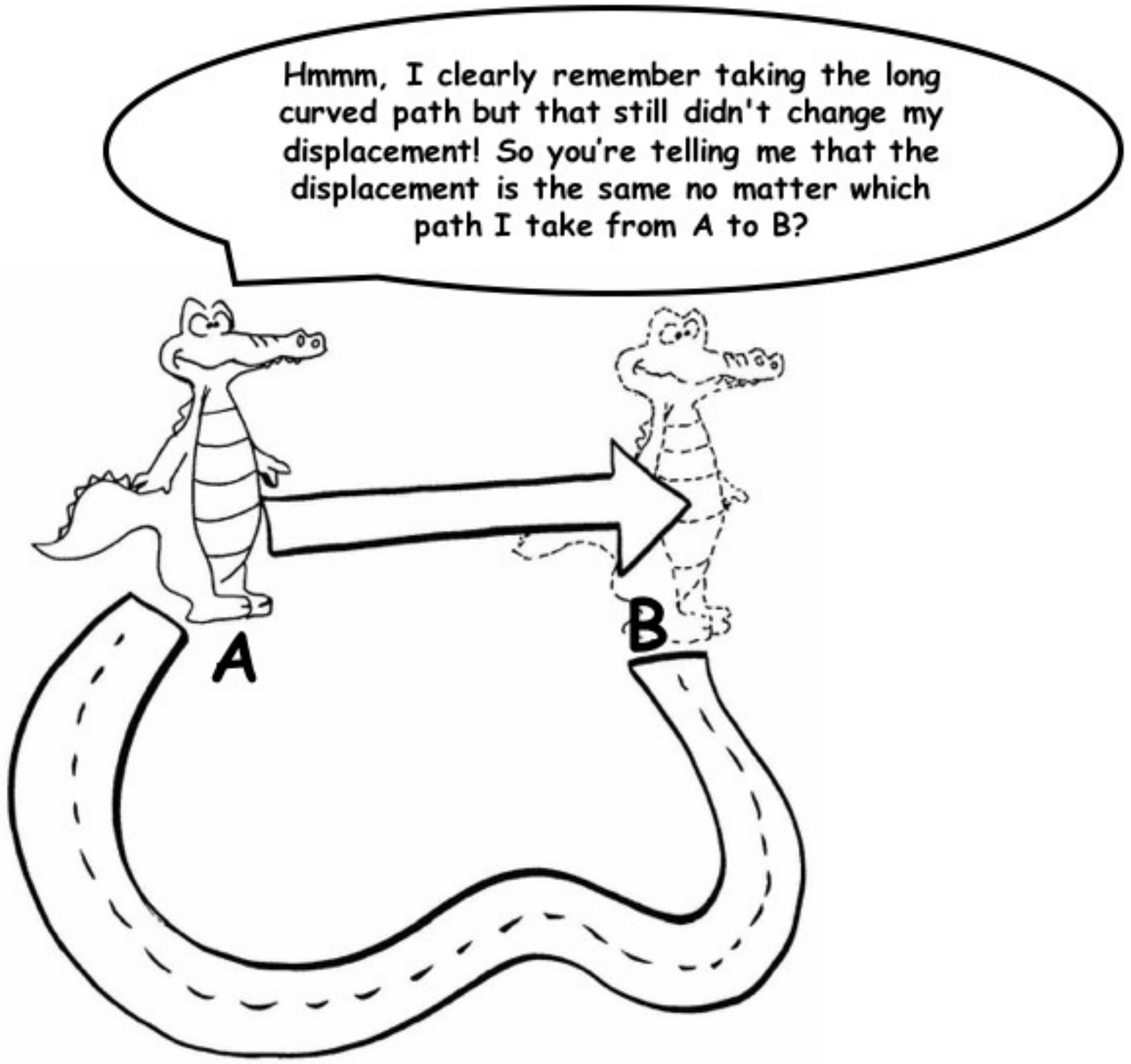
The Meter (m): The distance traveled by light in vacuum in 1/299792458.



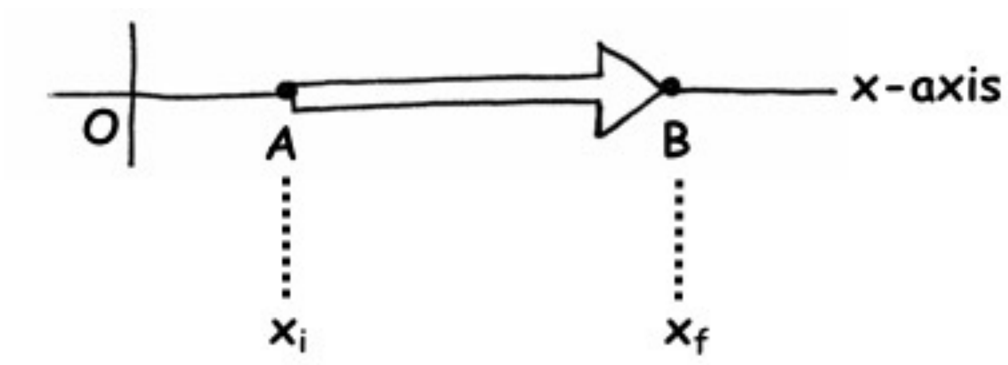
The Kilogram (kg): The mass of a specific platinum-iridium allow cylinder kept at the International Bureau of Weights and Measures.

The Second (s): 9192631770 periods of the radiation produced by cesium-133 atoms.

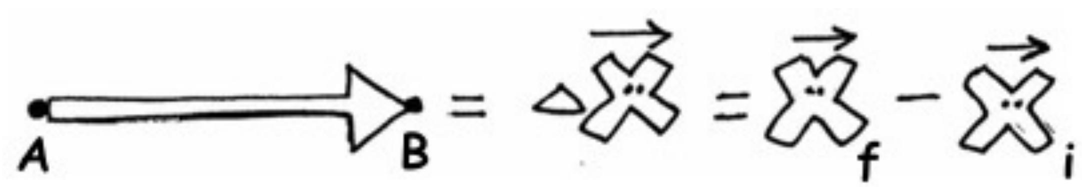
Displacement: An alligator decided to move from point A to B (where it's sunnier and more relaxing for it to stretch its muscles) along the long curved dotted line shown. The length of this path represents the total distance traveled by the alligator which is a scalar quantity. The alligator's displacement, on the other hand, is a vector quantity directed from A to B with a magnitude that is equal to the shortest distance from A to B. Like distance, the unit of displacement is meter (m).



If the displacement is along the x-axis

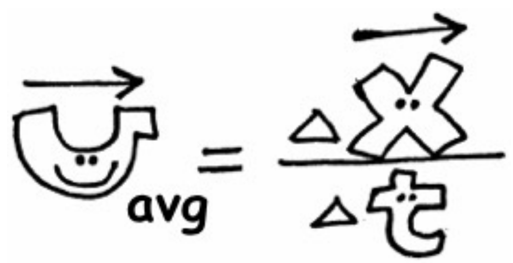


then we may write

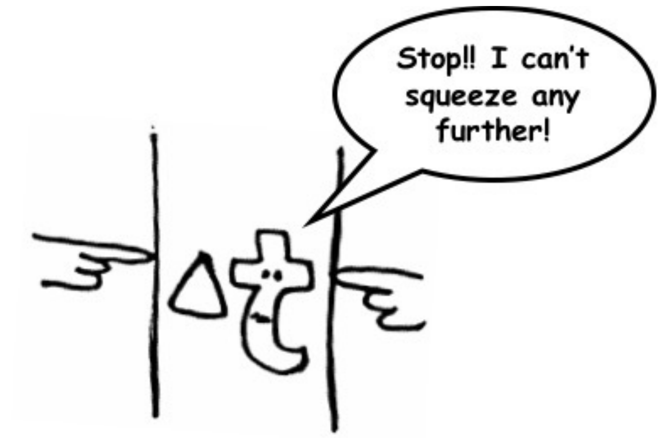


As you can see, the displacement depends only on the initial and final positions

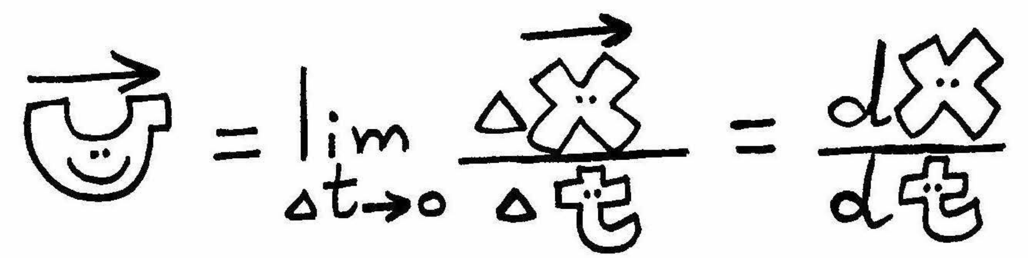
Velocity:



Average velocity is a vector quantity defined as the displacement divided by the time interval in which that displacement took place and it has the same direction as the displacement. Its unit is m/s. Now suppose that we choose Δt to be so small (by squeezing it) that it approaches ($\Delta t \to 0$),

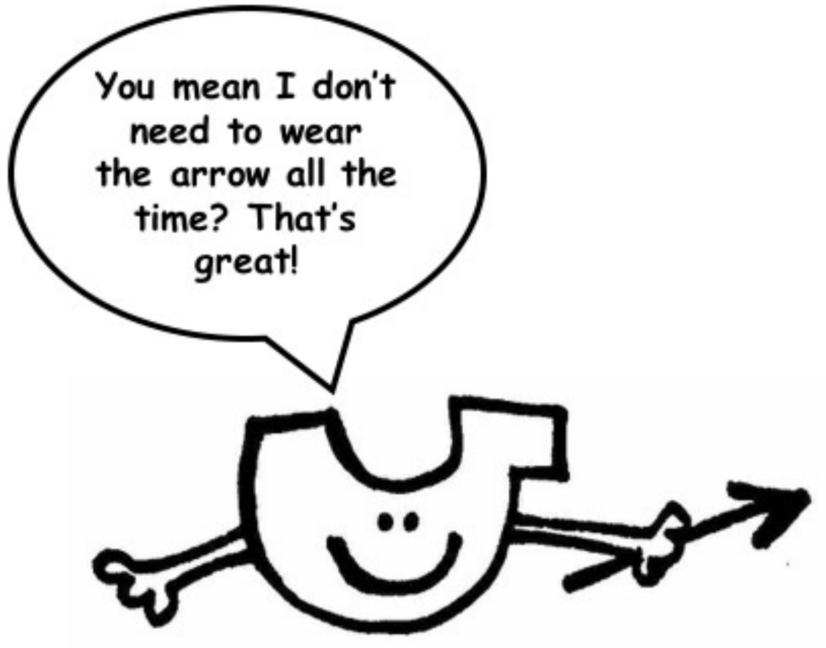


the velocity is then known as the instantaneous velocity where it represents the velocity at a particular instant of time and the subscript (avg) may be dropped



Unlike the average velocity which describes the overall motion of the object in a certain time interval, the instantaneous velocity allows for a more detailed description at particular instants of time.

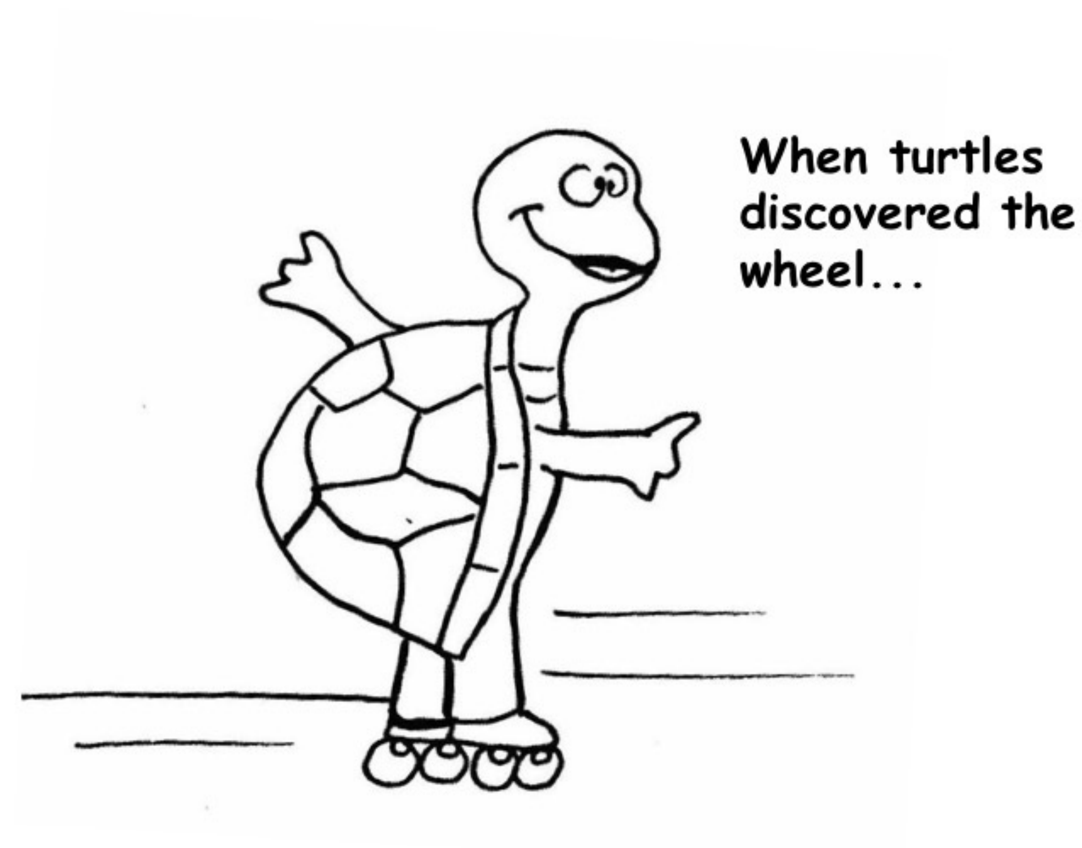
Speed: The magnitude of the velocity (either instantaneous or average) is known as speed written just as v without the arrow



Average Speed: is a scalar quantity that is different from speed in that unlike speed which is defined in terms of displacement, the average speed is defined in terms of the total distance traveled (that long path taken by the alligator) as follows

$$Average\ Speed = \frac{Total\ Distance\ Traveled}{Total\ Time}$$

Humans love speed! faster cars, faster planes, faster computers etc. Speed is associated with power, efficiency and the ability to save time. It can also be exhilarating, and turtles are no exception



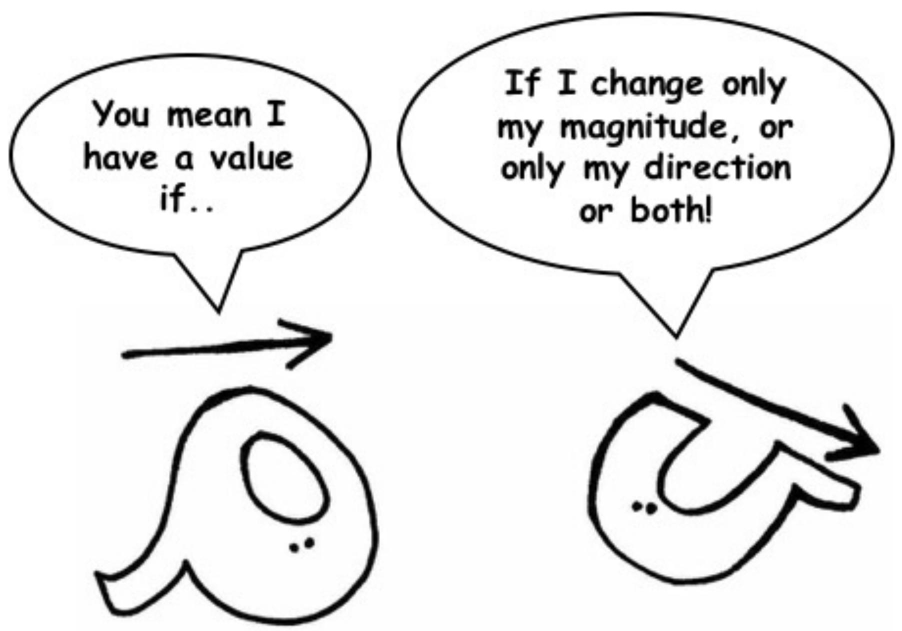
Acceleration: Is a vector quantity where average acceleration is defined as the change in velocity divided by the time in which that change took place. Its unit is m/s^2

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

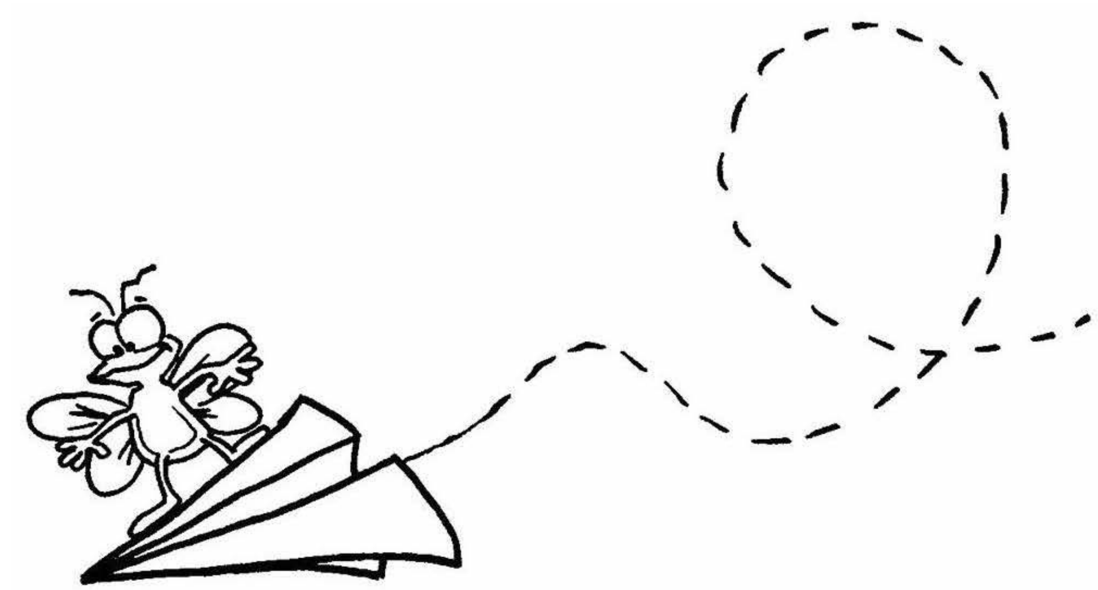
As in the case of velocity, the instantaneous acceleration is found when $\Delta t \rightarrow 0$ as

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Note that the acceleration is defined in terms of the velocity's change in both its magnitude (speed) and direction

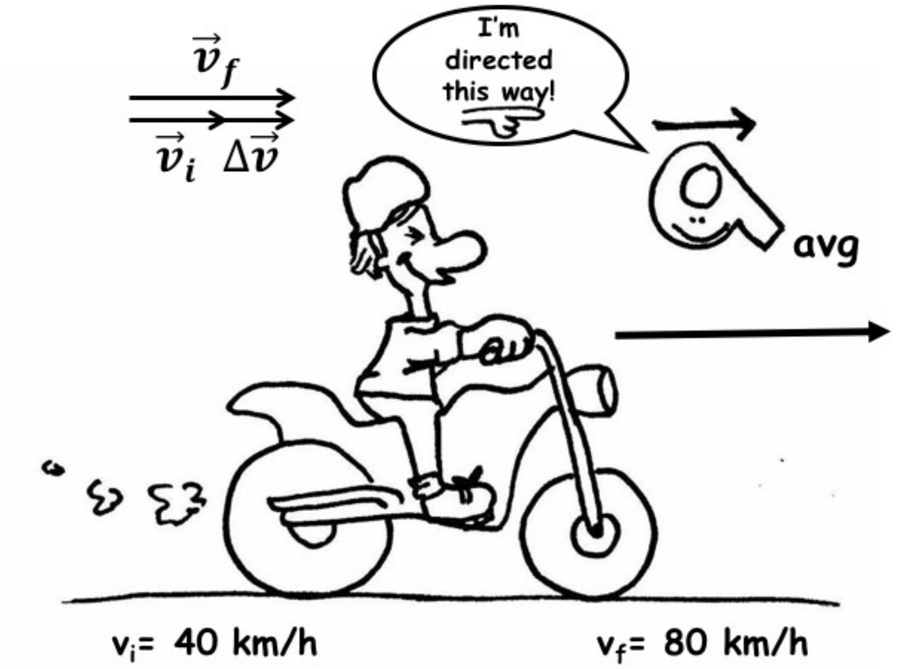


Motion in one dimension with an increasing or decreasing speed is an example of $\vec{v}$ changing in magnitude only (let's call it case 1). A uniform circular motion is an example of when $\vec{v}$ maintains its magnitude (speed) while changing its direction continuously (case 2), and a general motion along a curved path (such as the motion of this paper plane that was accompanied by a fly) is an example of when both the magnitude and direction of $\vec{v}$ change (case 3).

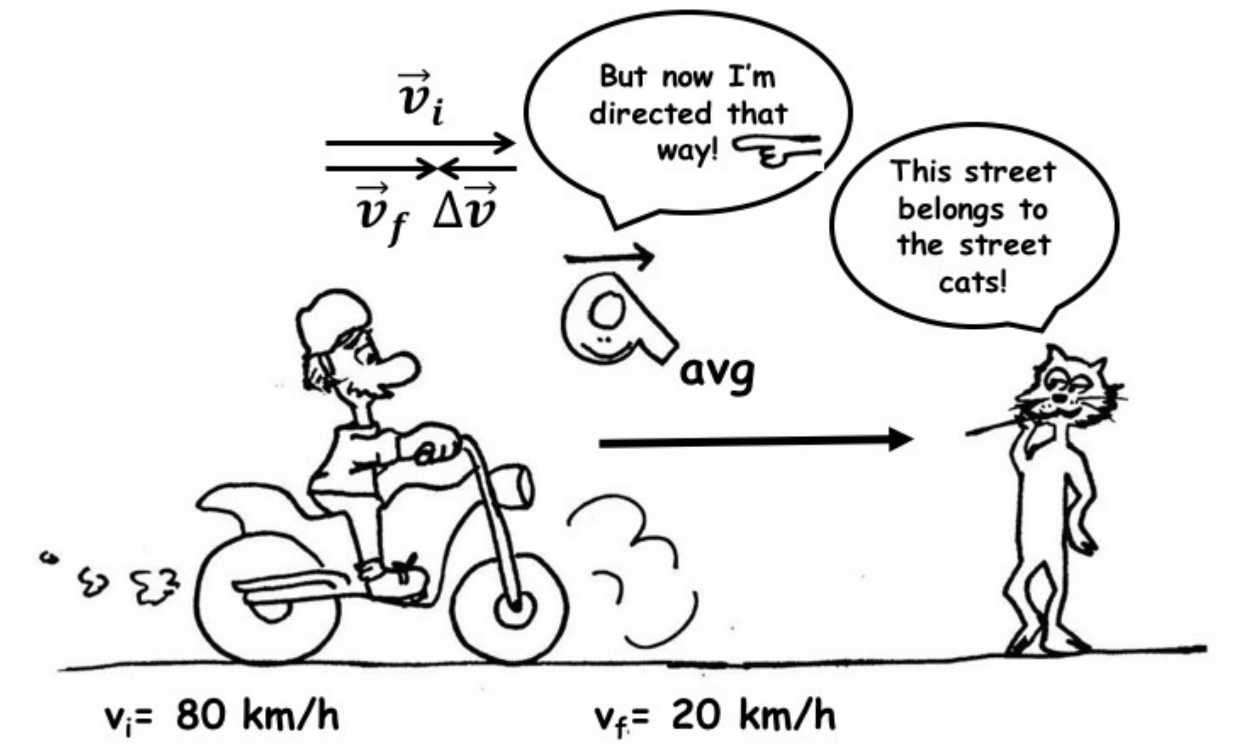


Note that the direction of the acceleration $\vec{a}$ is not necessarily in the same direction of the velocity $\vec{v}$.

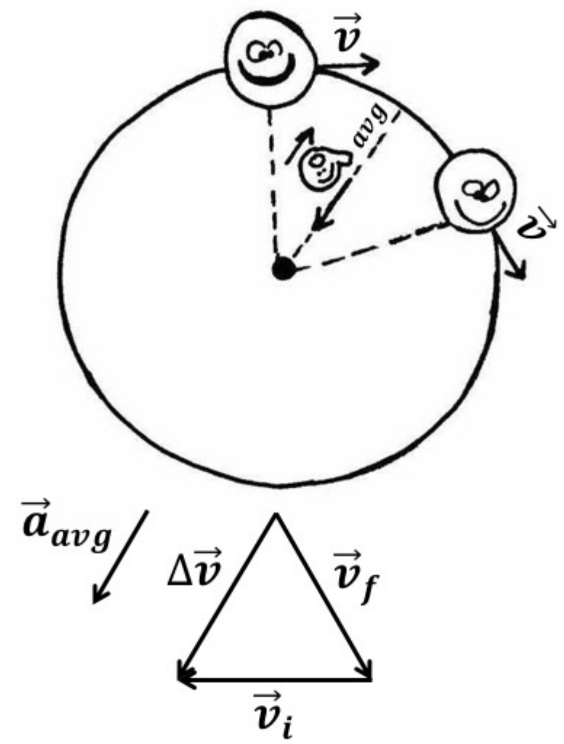
Let's consider an example of case 1 where Danny is moving along a straight line and accelerating from a speed of 40 km/h to 80 km/h during a time interval Δt. The average acceleration is then parallel to $\vec{v}$, in the same direction as the change in the velocity vector ($\Delta\vec{v}$) as shown.



Danny then sees a cat in front of him that is refusing to move and so he decelerates from a speed of 80 km/h to a speed of 20 km/h. In this case, the average acceleration's direction is opposite to the motion (antiparallel to $\vec{v}$).

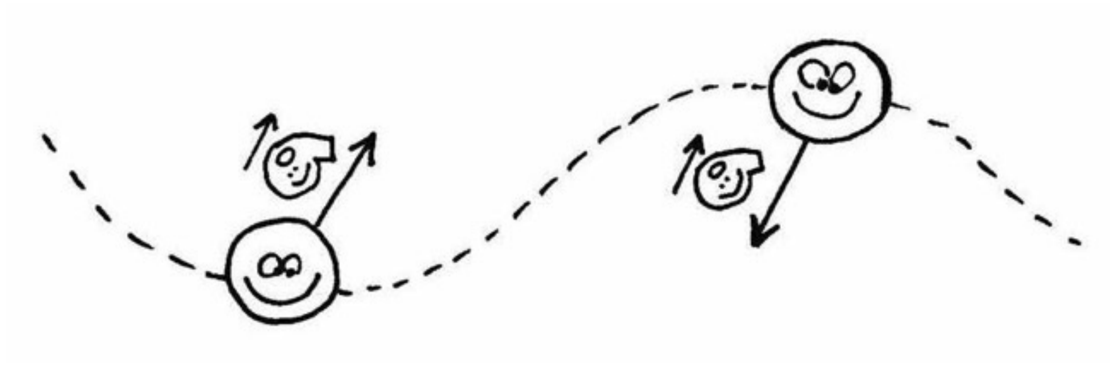


Now let's consider case 2. As you can see below, even though the speed is maintained constant and only the direction of $\vec{v}$ is changing, $\Delta\vec{v}$ does not vanish and is always perpendicular to $\vec{v}$ at each point towards the center of the circle. $\vec{a}_{avg}$ is in the same direction as $\Delta\vec{v}$ towards the center and therefore it is known as the centripetal (or normal) acceleration.



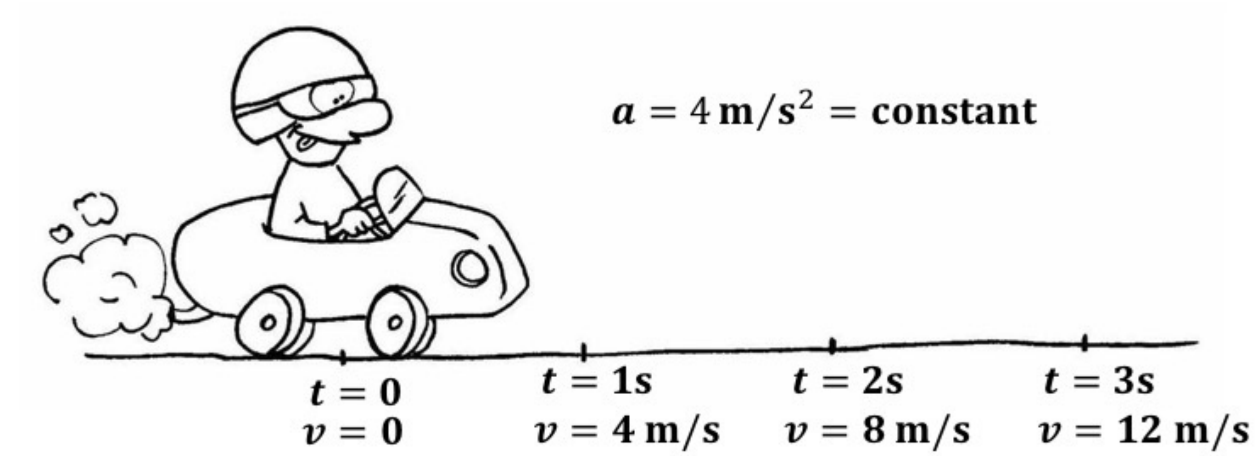
In case 3, where both the magnitude and direction of $\vec{v}$ are changing, the acceleration has two components, $\vec{a}_T$ corresponding to the change in the magnitude of $\vec{v}$ and $\vec{a}_N$ corresponding to the change in the direction of $\vec{v}$. The total acceleration is the vector sum of the component vectors and is at some angle to the path at each point as shown.

$$\vec{a} = \vec{a}_T + \vec{a}_N$$



One-Dimensional Motion with Constant Acceleration:

In this type of motion, the average acceleration during any time interval is equal to the instantaneous acceleration at any instant of time and the velocity will change linearly with time. I.e., the velocity will increase (or decrease) by the same amount each second like this race car that starts from rest and accelerates at a constant rate of 4m/s^2.



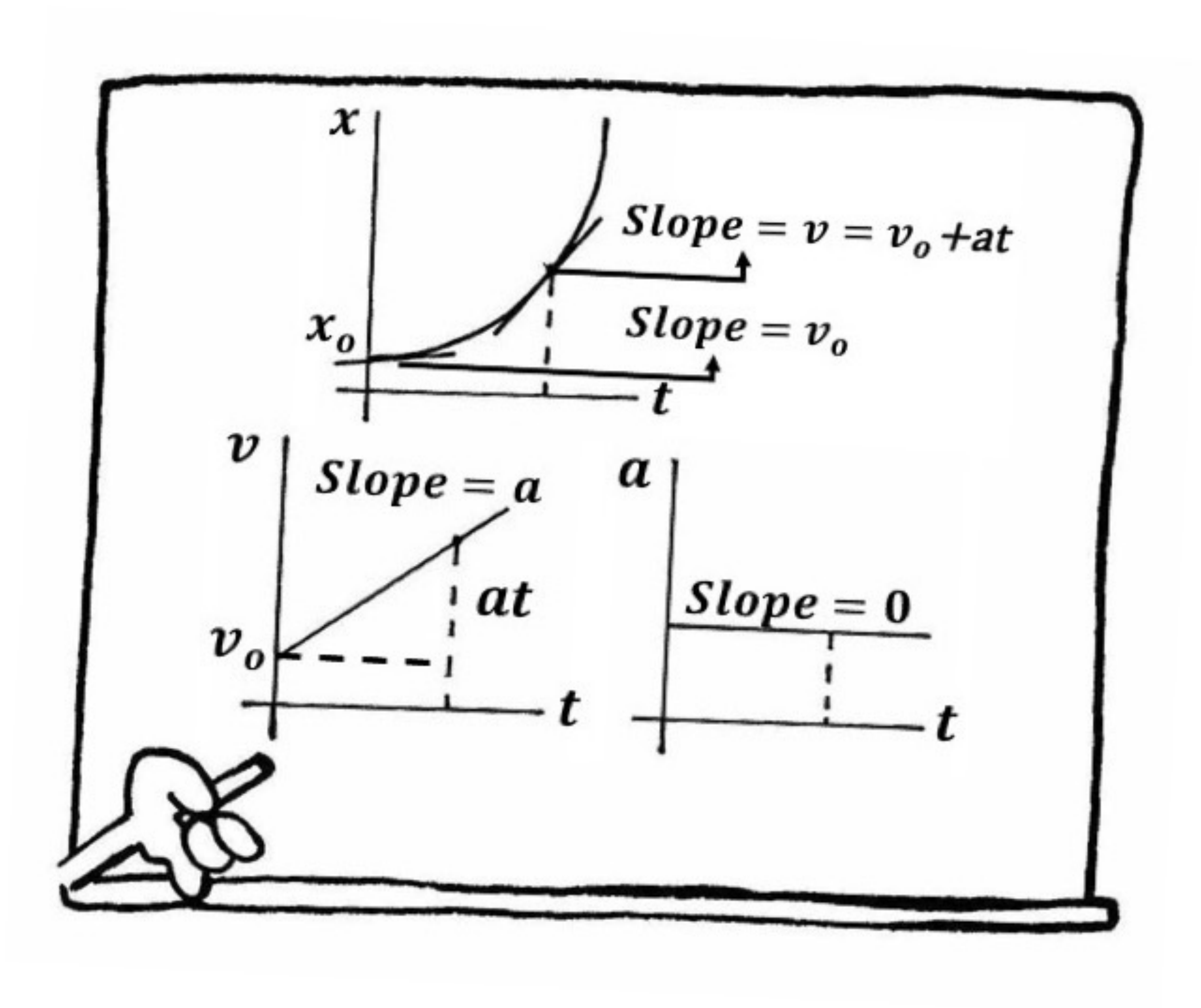
A simple derivation gives the four famous kinematic equations for such motion. Setting $t_i = 0, t_f = t, v_{xi} = v_o, v_{xf} = v,\ x_i = x_o$ and $x_f = x$, the average velocity can be expressed as

$$\bar{v} = \frac{(v + v_o)}{2} = \frac{\Delta x}{\Delta t}$$

from this you can find that

$$x - x_o = \frac{1}{2}(v + v_o)t \quad (1)$$
$$v = v_o + at \quad (2)$$
$$x - x_o = v_o t + \frac{1}{2}at^2 \quad (3)$$
$$v^2 = v_o^2 + 2a(x - x_o) \quad (4)$$

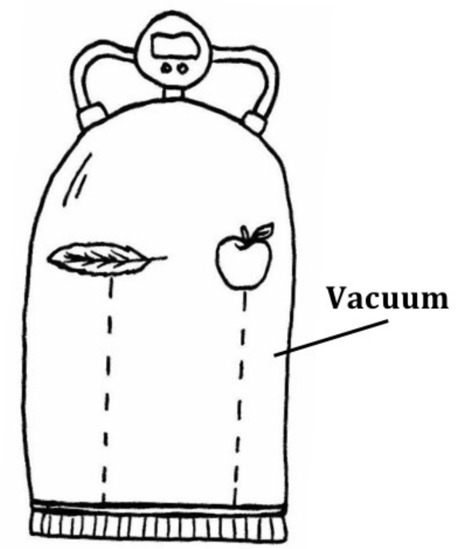
This is the graphical representation of this motion



Free Fall:

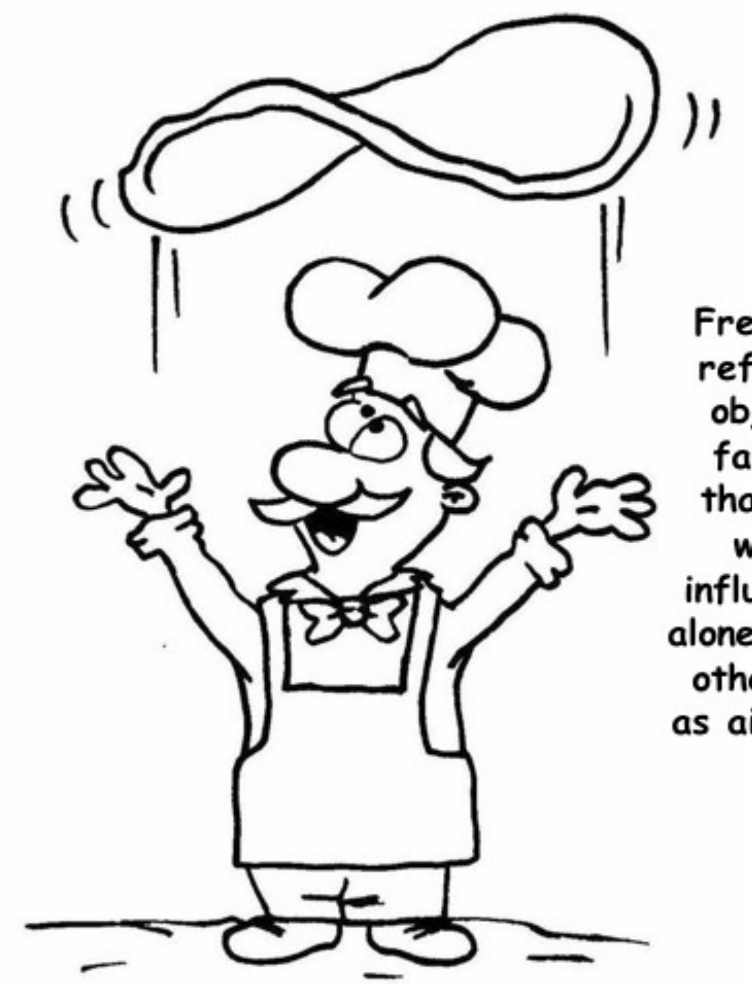
By dropping different objects from the leaning Tower of Pisa (or experimenting with objects moving on inclined planes according to another story), Galileo Galilei studied the acceleration of falling bodies and discovered that if air resistance is neglected, all objects, regardless of their mass or size, would fall with the same constant acceleration g = 9.8 m/s^2 that is due to Earth's gravity.

This means that if an apple and a feather are released from rest in a vacuum chamber, they will simultaneously hit the ground.



This was a surprise that woke up this bird from his sleep

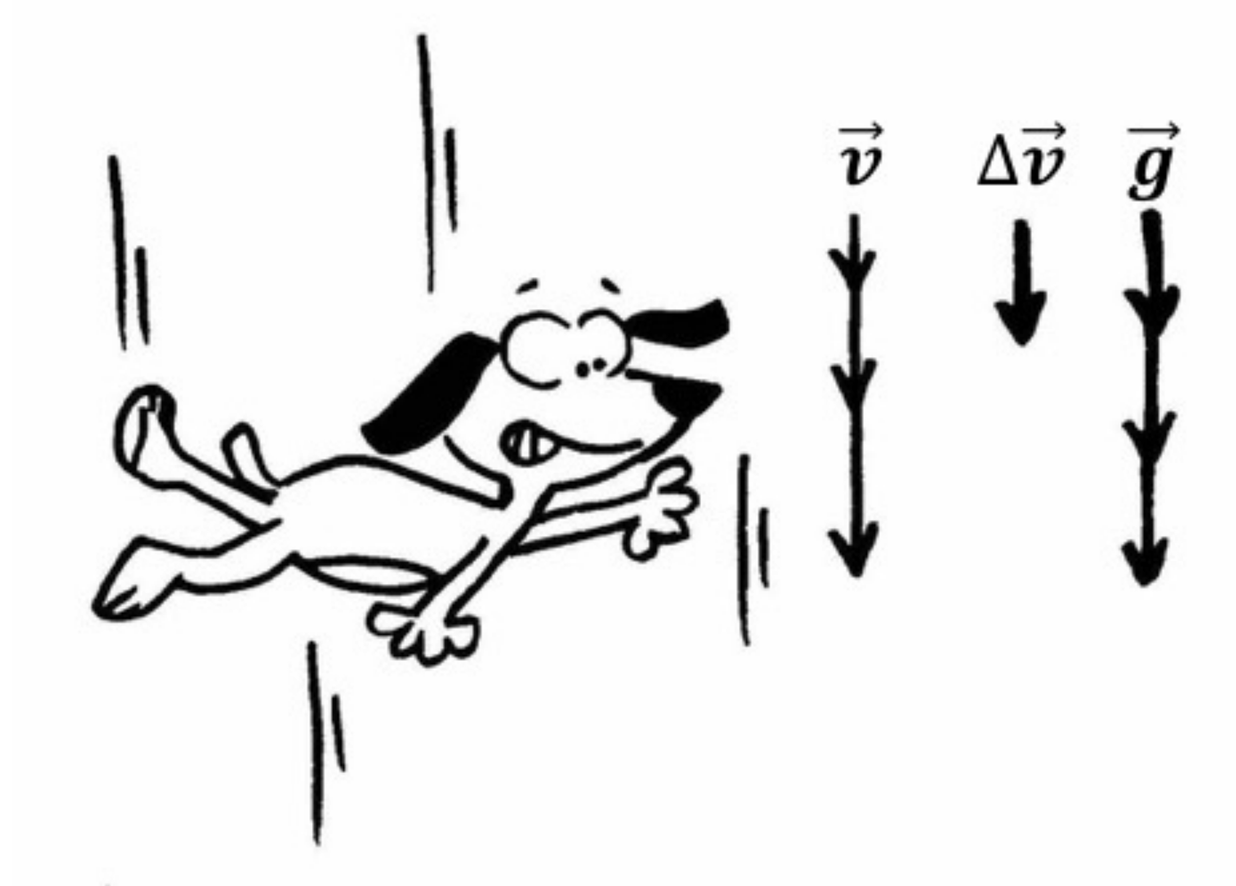
The value of the gravitational acceleration g varies with altitude as well as other factors (discussed later), but near the Earth's surface, it can be assumed to have a constant magnitude of 9.8 m/s^2. The direction of the vector $\vec{g}$ is towards the center of the Earth.



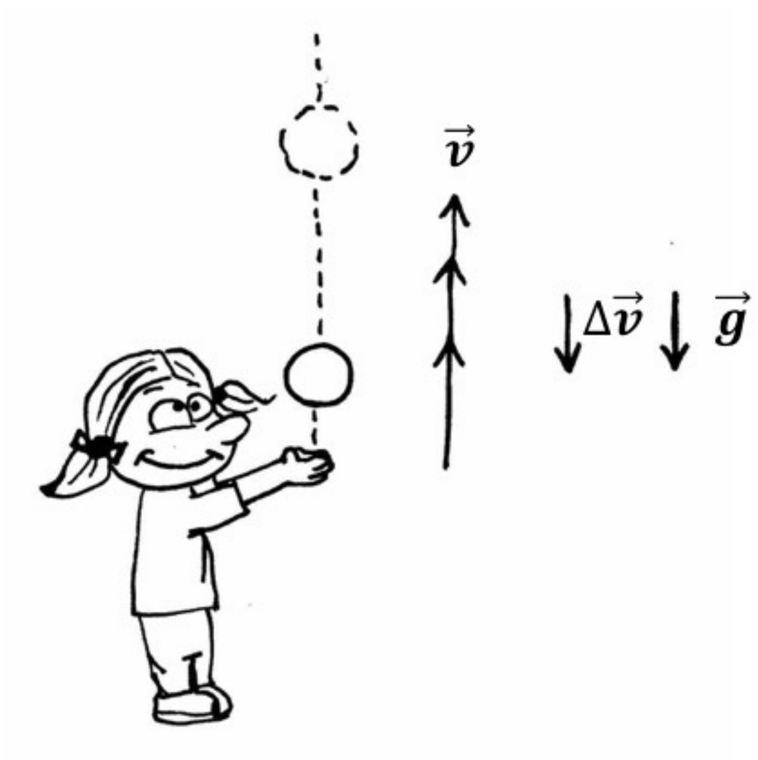
Free falling motion refers to not only objects that are falling but those that are rising as well under the influence of gravity alone and in which all other factors such as air resistance are neglected.

Free falling motion is an example of motion along a straight line with a constant acceleration. The kinematic equations can be found by simply replacing x by y and a by $-g$ when the positive y direction is taken to be upwards.

Consider Bud who was practicing for his gymnastics tournament when he fell on the mat which absorbed his impact. As he fell, his speed increased steadily at the constant rate of g = 9.8 m/s^2 as shown by the increasing velocity vector, while the acceleration's vector is constant throughout the motion.



If a ball is thrown upwards, its speed will decelerate at the constant rate of g = 9.8 m/s^2 until it reaches zero at the maximum height where it then falls back and accelerate.



Two-Dimensional Motion:

In treating two-dimensional motions, the motion in each of the x and y directions can be considered independently. The position, velocity and acceleration vectors are

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

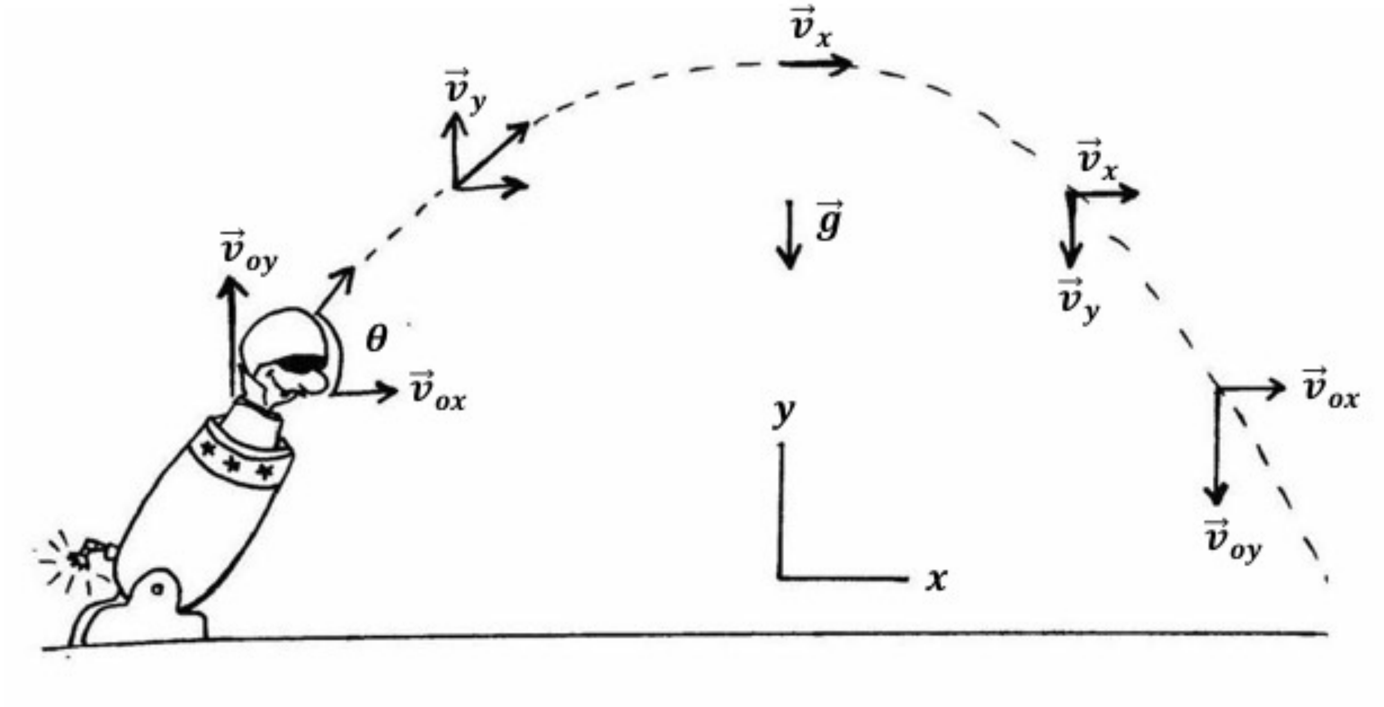
$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$$

Where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in each of the x and y directions respectively.

In a simplified model where air resistance and the Earth's curvature and rotation are neglected, and if the free-fall acceleration g is constant in magnitude and direction throughout the motion, the path of the projectile is then always a parabola that is determined by the magnitude and direction of its initial velocity. Since the x and y motions can be treated separately, the projectile motion can be considered as a combination of a horizontal motion with constant velocity (zero acceleration) and a vertical motion with a constant acceleration g directed downwards.

Let's consider Eric, a very adventurous guy, who insisted on becoming a human cannonball. When fired, the y component of his velocity v_y will behave as in a free fall motion, first it will decrease as he moves up, reaches zero at the maximum height and then increases again as he moves down. The x component of velocity v_x will remain constant throughout the motion since there is no acceleration in that direction.

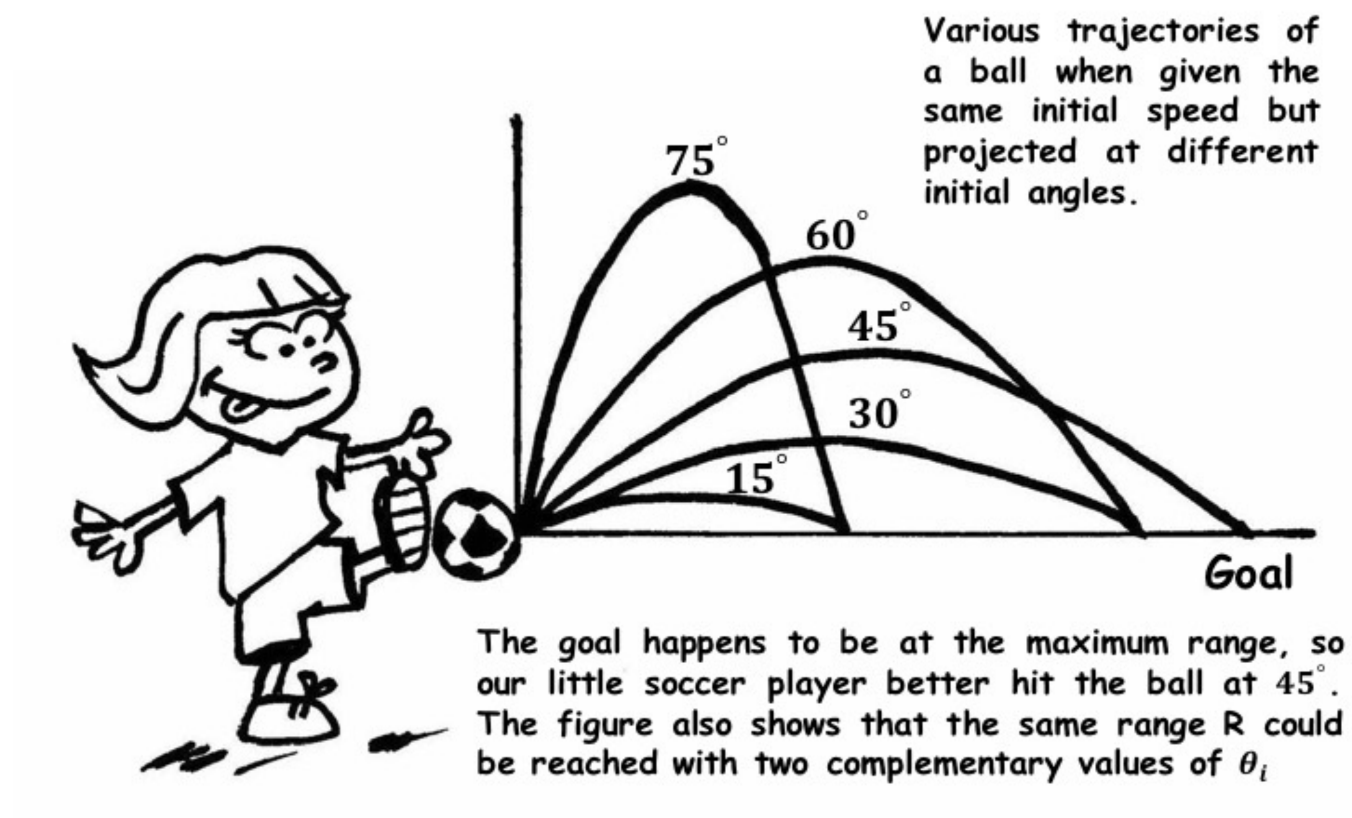


Through some simple derivation, you can show that the maximum height h and the maximum horizontal range R reached by any projectile is given by

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

$$R = \frac{v_i^2 \sin 2\,\theta_i}{g}$$

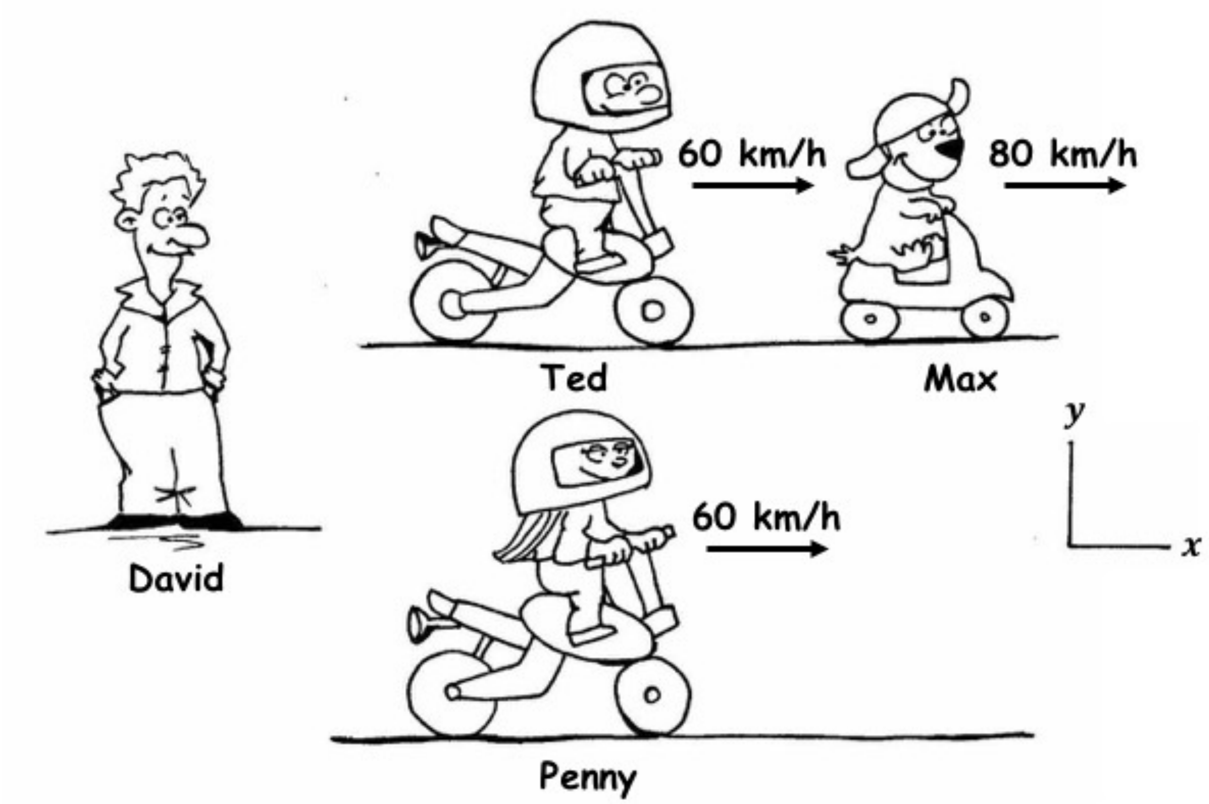
This shows that the maximum range $R_{max} = \frac{v_i^2}{g}$ is when $\sin 2\theta_i = 1$ which gives $2\theta_i = 90^\circ$ and $\theta_i = 45^\circ$.



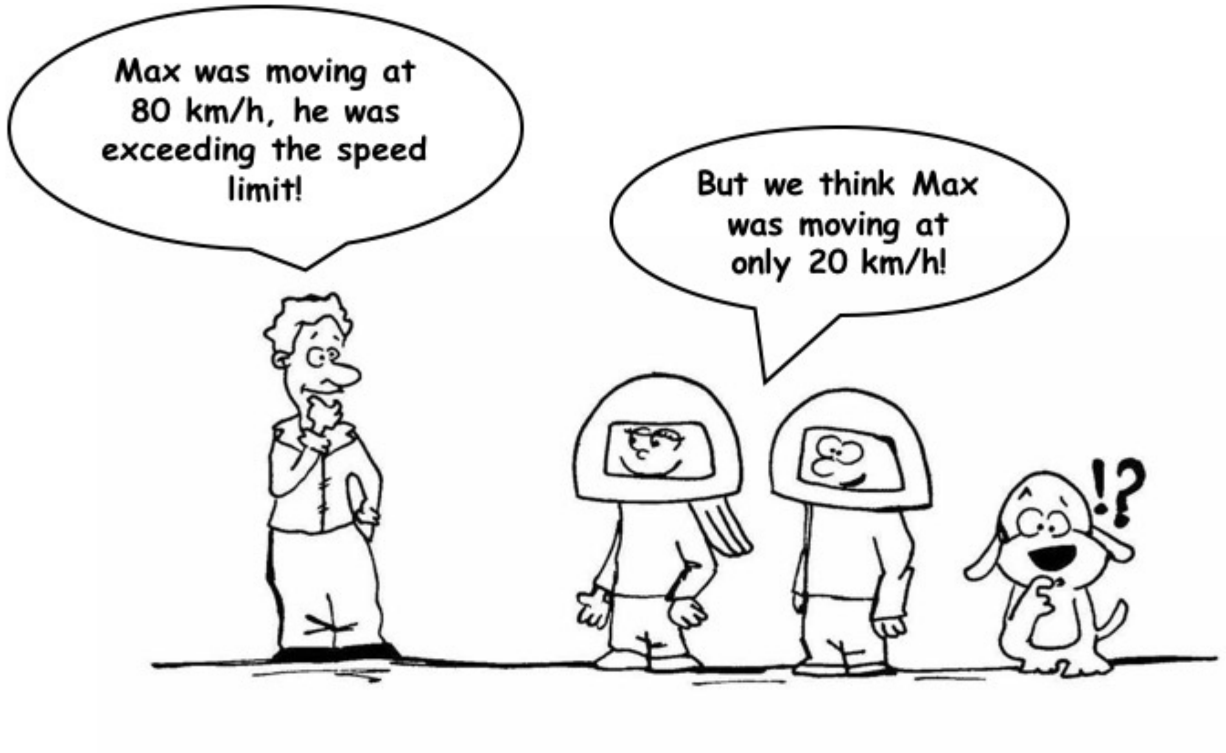
Relative Velocity:

Imagine being on a train that is next to another train where from your window there is only it and no other frame of reference. If that train moves forward, it can seem as though you are moving backwards and vice versa. This can give you a rough idea of how the description of motion depends on the observer (but the laws of physics are the same as we shall see later).

Consider Ted and Penny, who went out for a ride on their motorcycles with their pet Max, who is leading the way. David, their friend who is at rest relative to the ground, noticed that Max is exceeding the speed limit of 60 km/h. Since both Ted and Penny are moving at 60 km/h, they are at rest relative to each other and moving at 60 km/h relative to David. According to both of them, David is moving backwards in the negative x-direction at -60 km/h. Furthermore, Although Max is moving at 80 km/h relative



to David, he is moving at (80 (km/h)-60 (km/h))= 20 km/h relative to either Penny or Ted



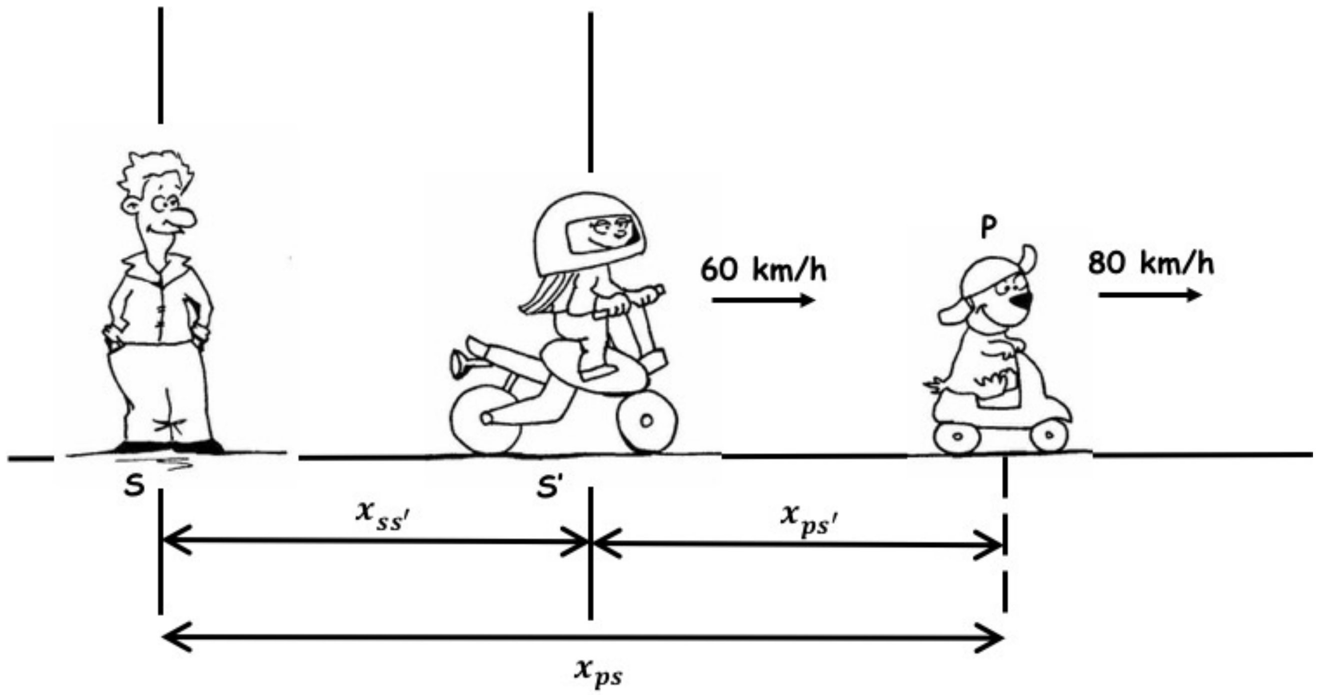
This confusion was sorted out by Galilean transformation. Suppose we attach a coordinate system to each of David and Penny, then each is said to be in a reference frame.

From the figure, you may see that

$$x_{ps} = x_{ps'} + x_{s's}$$

Differentiating with respect to time gives

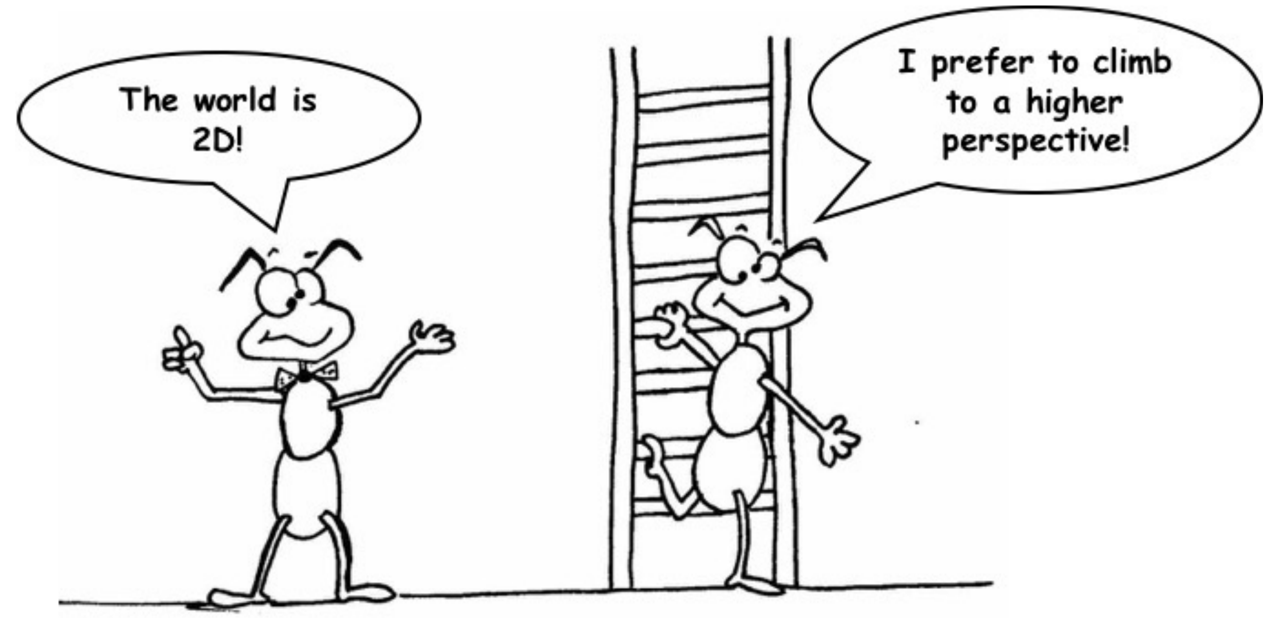
$$v_{ps} = v_{ps'} + v_{\acute{s}s}$$



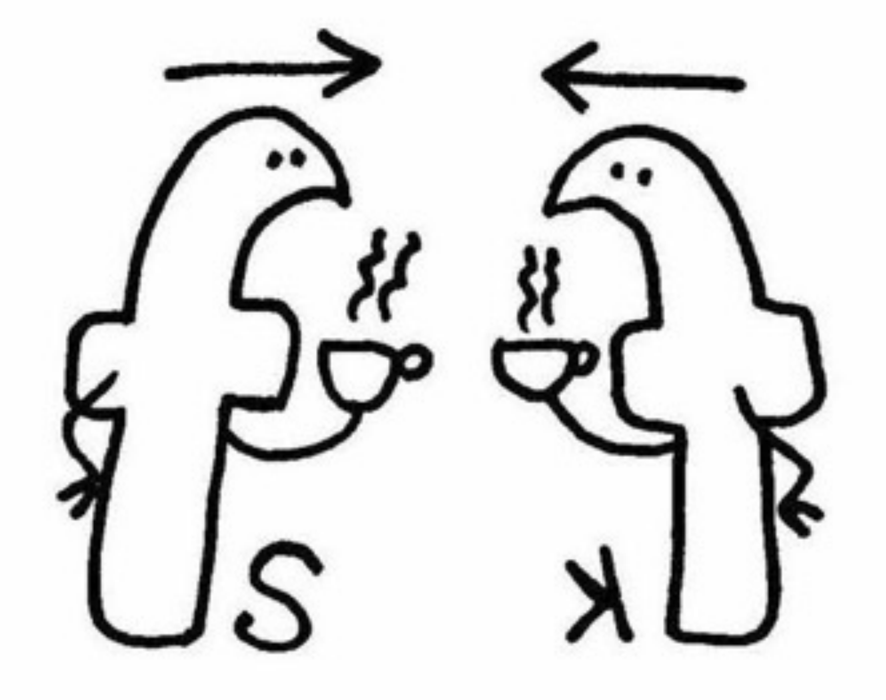
Note that v_{ps} is the velocity of Max relative to David, $v_{ps'}$ is the velocity of Max relative to Penny and $v_{s's}$ is velocity of Penny relative to David.

$$\Rightarrow \quad v_{ps'} = v_{ps} - v_{s's} = (80(km/h) - 60(km/h)) = \; 20\; km/h$$

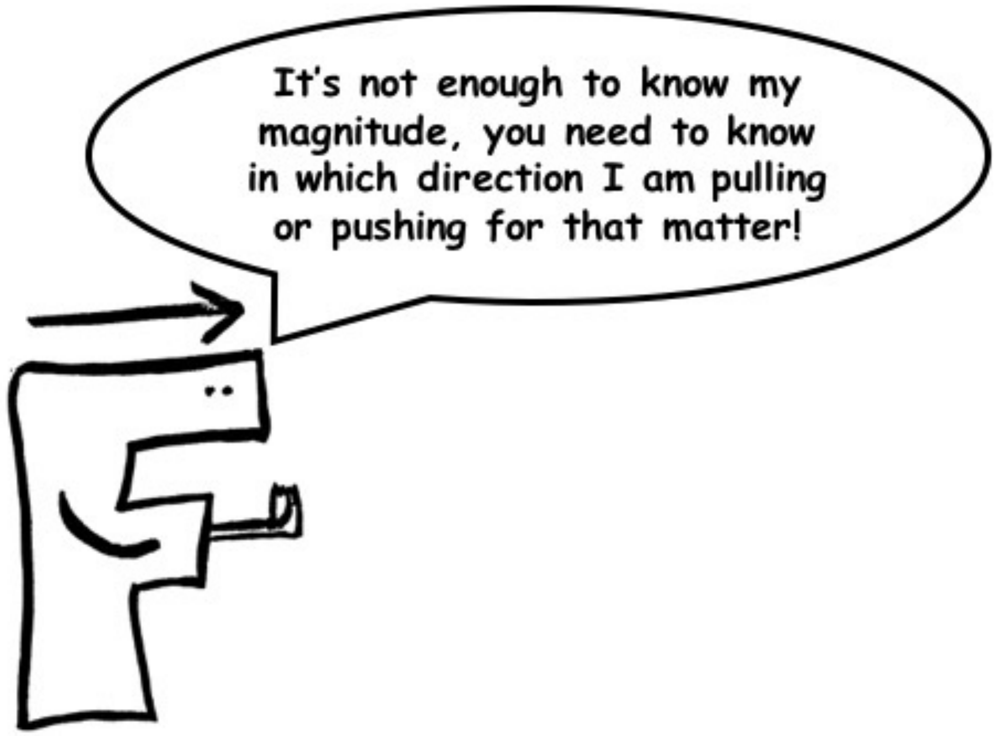
Therefore, the value of $\vec{v}$ depends on the reference frame. David and Penny can agree now and Max would have to slow down!



Newton's Laws

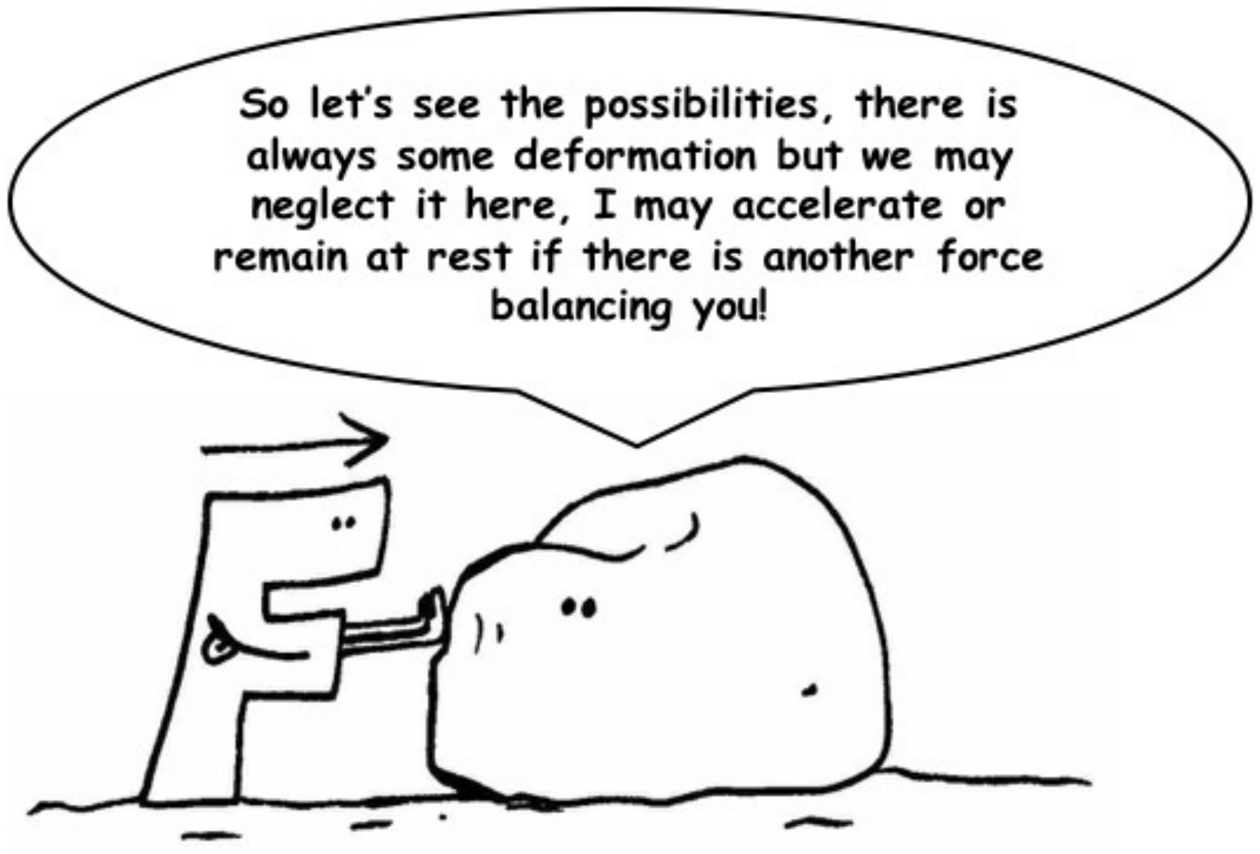


Before discussing Newton's laws, let's first define the concept of force. A force is a pull or a push in a certain direction resulting from the interaction between one object

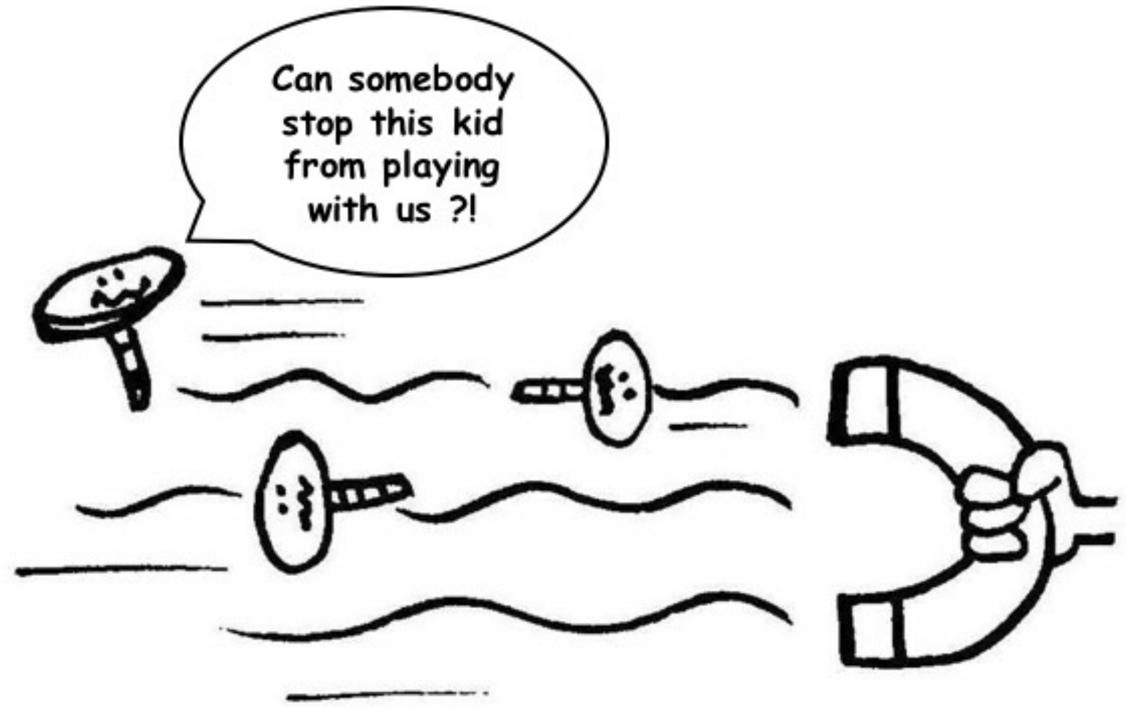


and anther or between an object and its environment. Therefore, force is a vector quantity as it is clearly aware of that.

A force can cause an object to accelerate or deform (or both). However, if the force is balanced with another force in the opposite direction, then the net external force acting on it is zero and it will not accelerate (either remain at rest or continue moving at a constant velocity).



Therefore, acceleration is a measure of the net force on the object. There are two kinds of forces: contact forces which involve direct contact between the objects (such as kicking a ball) and field forces in which physical contact is not necessary as they can act through empty space (such as the gravitational force or the magnetic force).

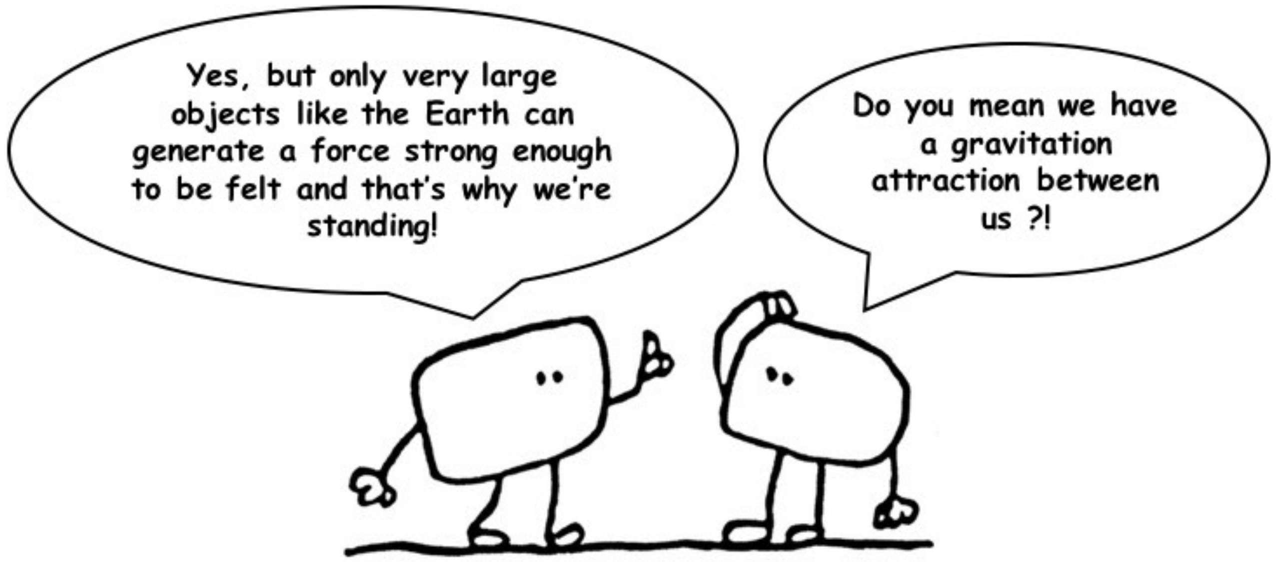


The magnetic force is an example of a field force

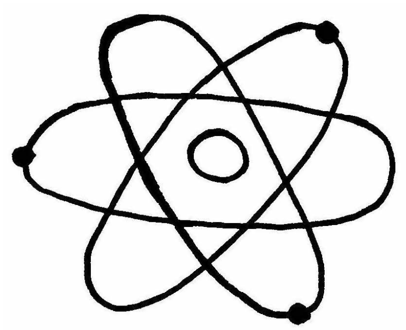
This distinction is convenient on the macroscopic scale, however at the atomic scale, contact forces are actually electric forces which are field forces.

There are four fundamental forces in nature so far discovered which are all field forces:

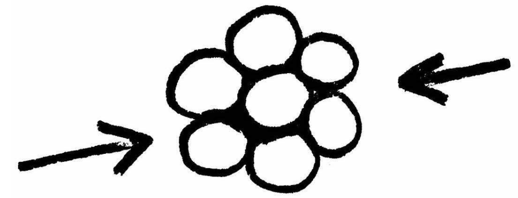
1- The gravitational force between any two objects



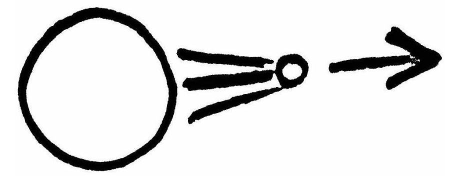
2- The electromagnetic force between electric charges, such as between the electrons and protons in an atom



3- The strong nuclear force between subatomic particles which binds the nucleus of an atom



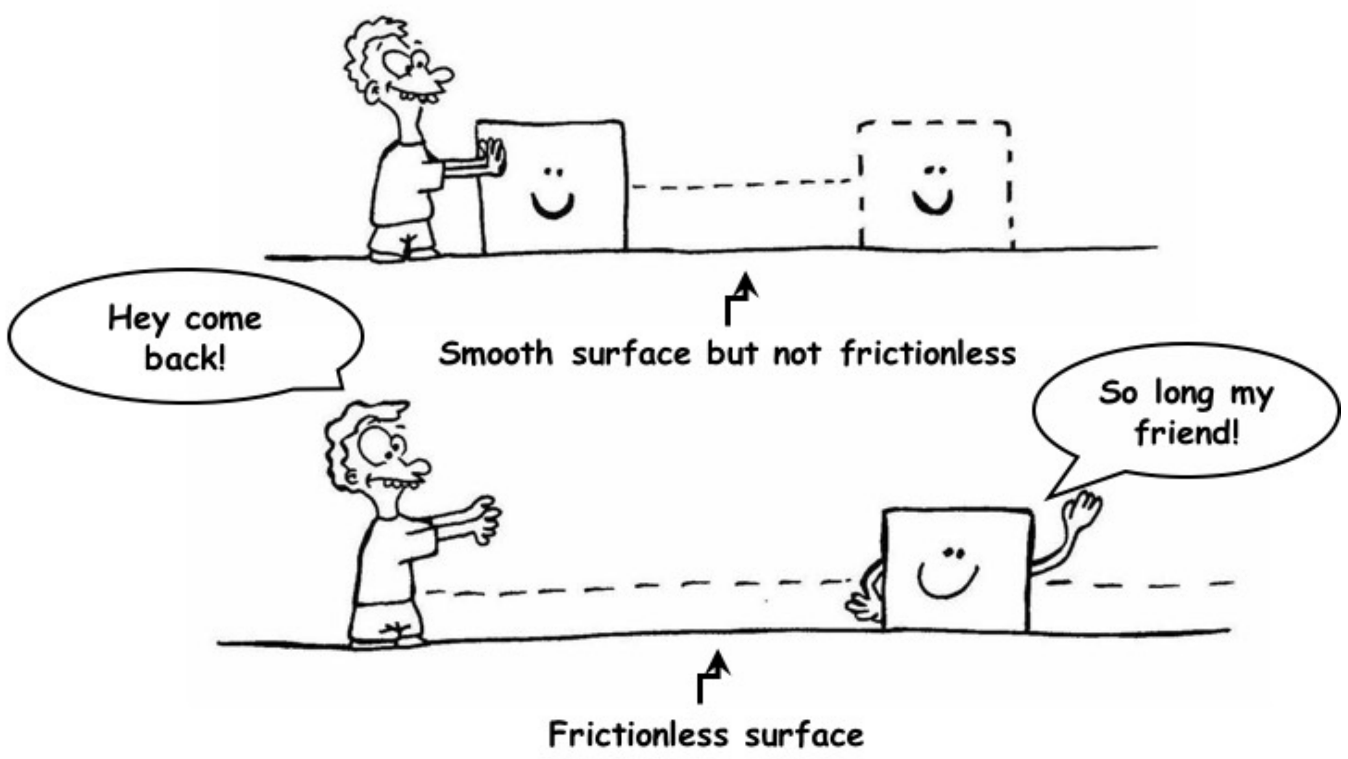
4- The weak nuclear force which causes radioactive decay and make some nuclei unstable



The first two forces are long-range forces and can act over large distances. The second two are short-range forces.

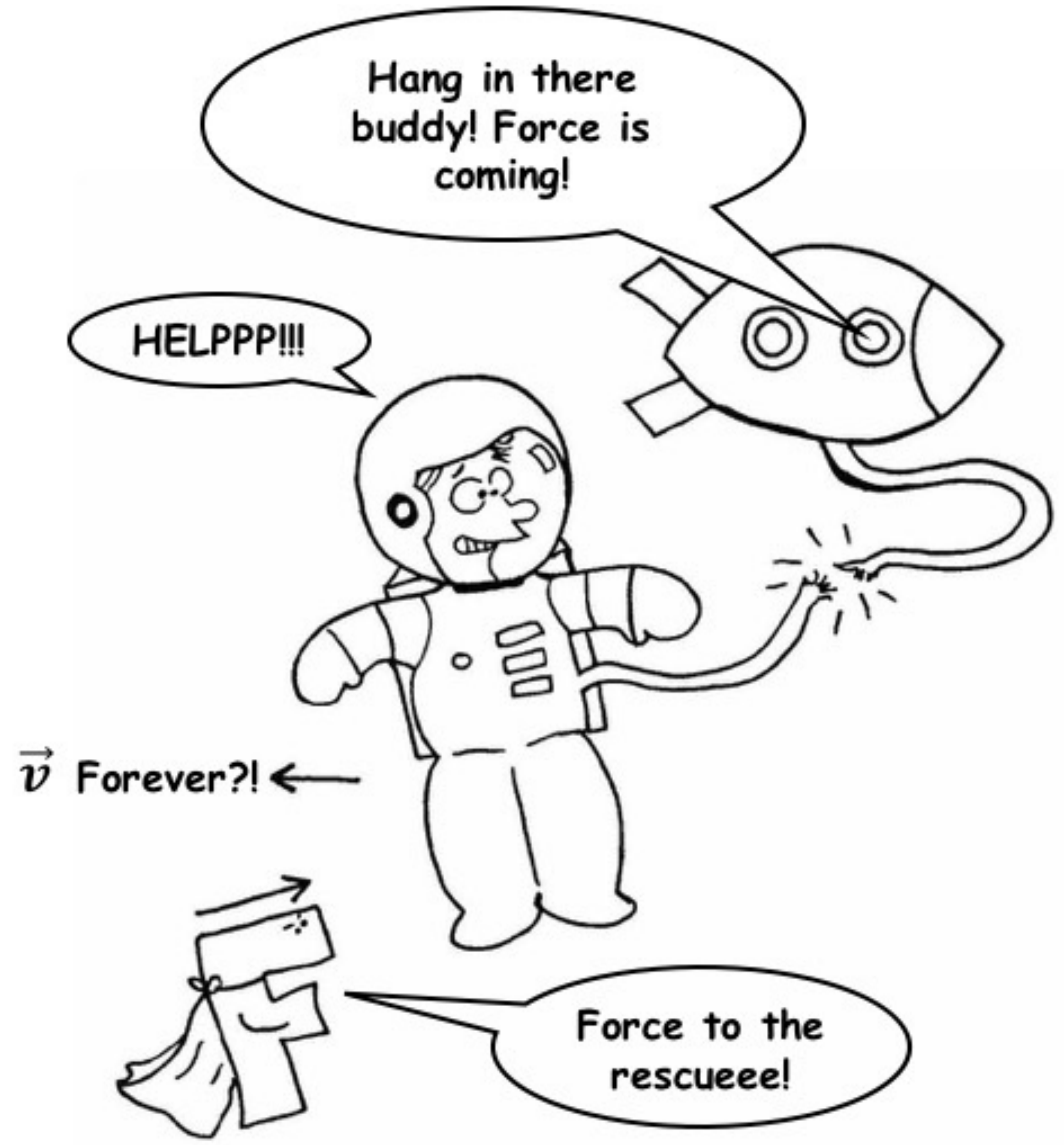
Newton's First Law:

Imagine giving a block a push as shown below. The block will move for some distance before it stops. As the surface gets smoother, this distance will increase. If the surface becomes frictionless, the block will continue to move along a straight line with a constant speed (constant velocity) without requiring any force to keep it moving.



This is summarized in Newton's first law as follow: *An object at rest will remain at rest and an object in motion will continue in motion with a constant speed along a straight line unless acted upon by a net external force*. This tendency to stay at rest or to keep in motion is known as inertia.

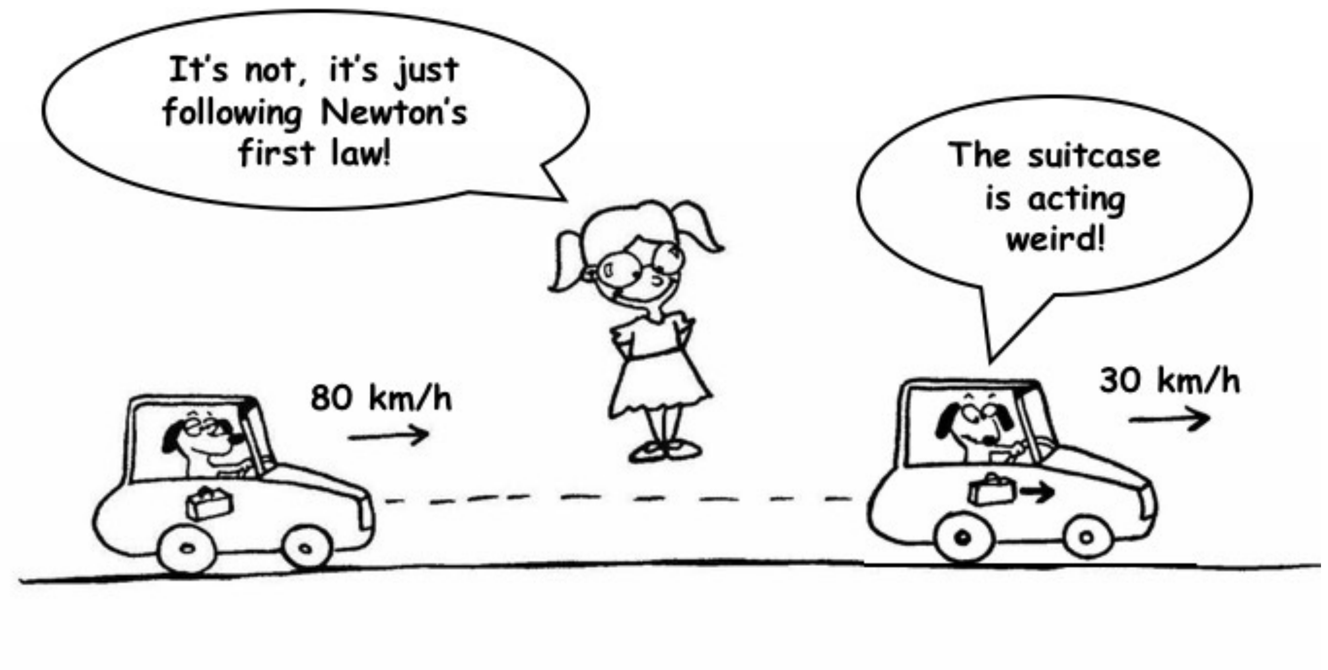
This is why an astronaut is tied to the ship with a strong rope when doing a spacewalk, since if the rope is cut, the astronaut will float away at a constant speed in the same direction forever unless a force acts on him/her such as the gravitational force of a planet. For safety astronauts are equipped with small jet thrusters that provide the force necessary to change their state of motion so that they can maneuver their way back to the ship.



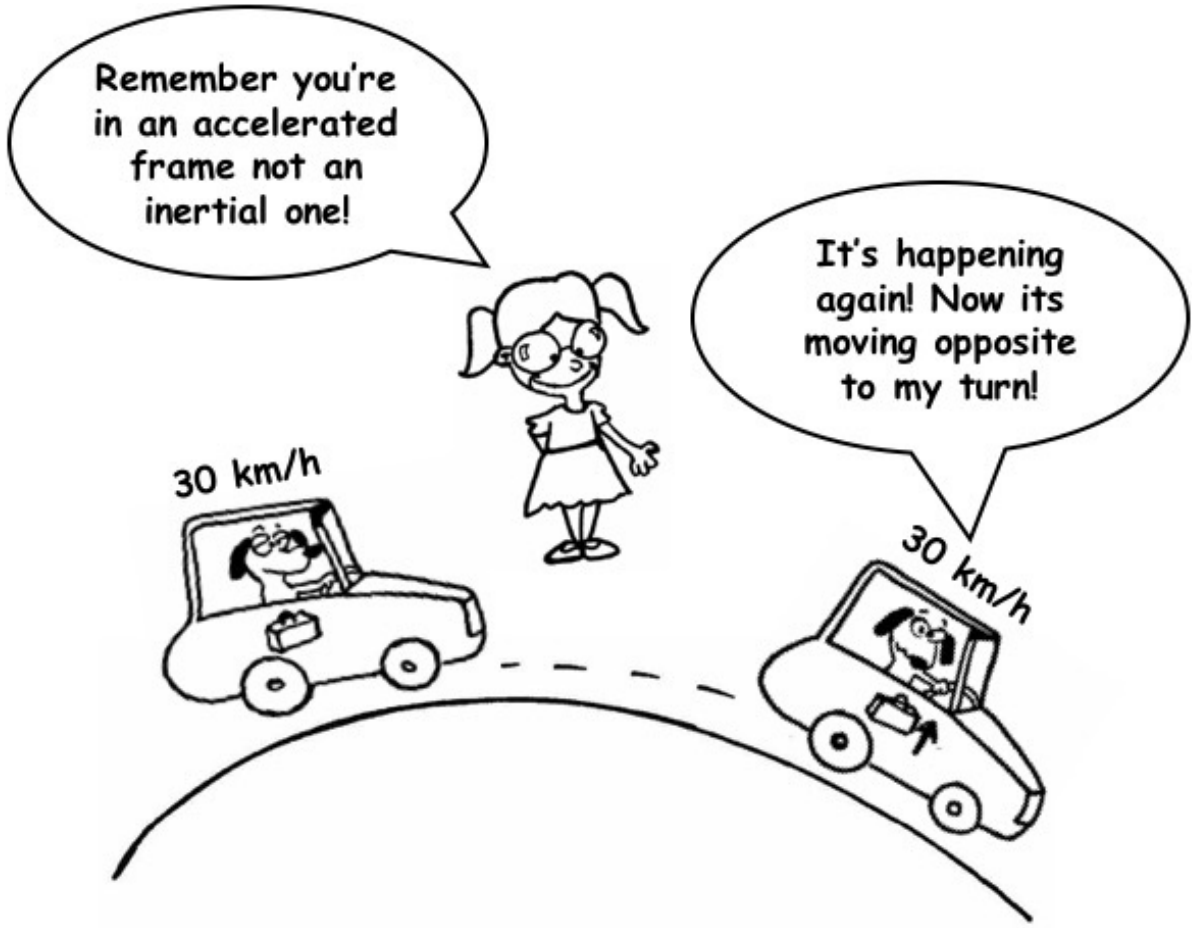
Inertial Frame of Reference:

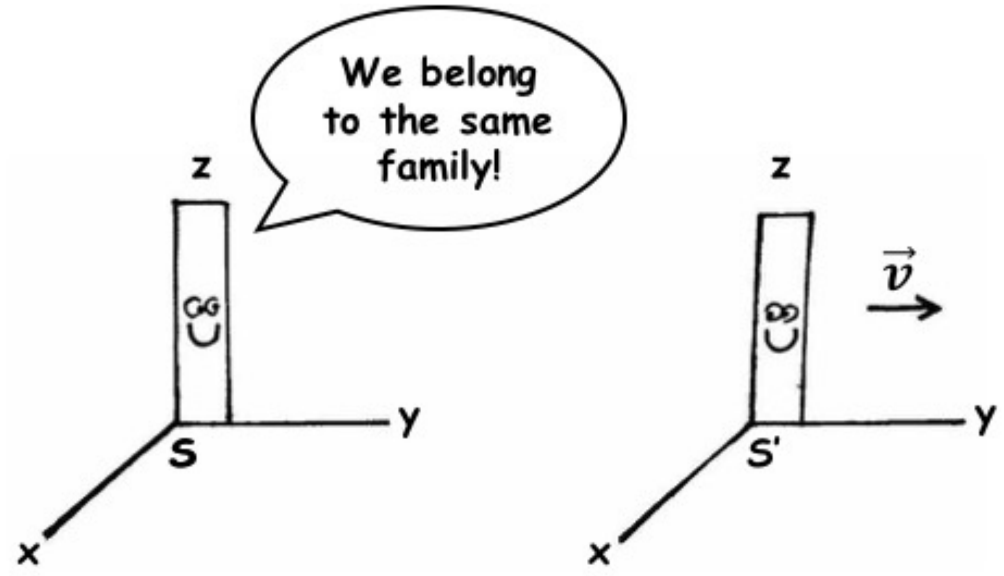
In an inertial frame of reference, Newton's first law is valid, i.e. if there is no net force acting on an object, then the object's acceleration is zero. This can be illustrated by the example shown below. Let's consider Bud who was driving his car back from work with his suitcase in the seat just next to him. When Bud (who was fastening his seatbelt) decelerated the car from a speed of 80 km/h to 30 km/h when near the traffic light, he noticed that the suitcase next to him started moving forward by itself without any apparent force acting on it.

When convinced there is nothing spooky, Bud (who is in an accelerated frame of reference) concluded that Newton's first law is invalid since the suitcase moved without an apparent force acting on it. The situation, however, is different to Maggie, who was observing Bud and wondering why he seemed concerned. Since Maggie is in an inertial frame of reference (not an accelerated frame), to her the suitcase was initially moving with a constant velocity and the net force acting on it was zero. When Bud decelerated the car, the net force acting on the suitcase was still zero and so it must continue to move along a straight line with a constant speed (constant velocity) and stop by friction or impact with the interior of the car. Therefore, according to Maggie, Newton's first law is valid.

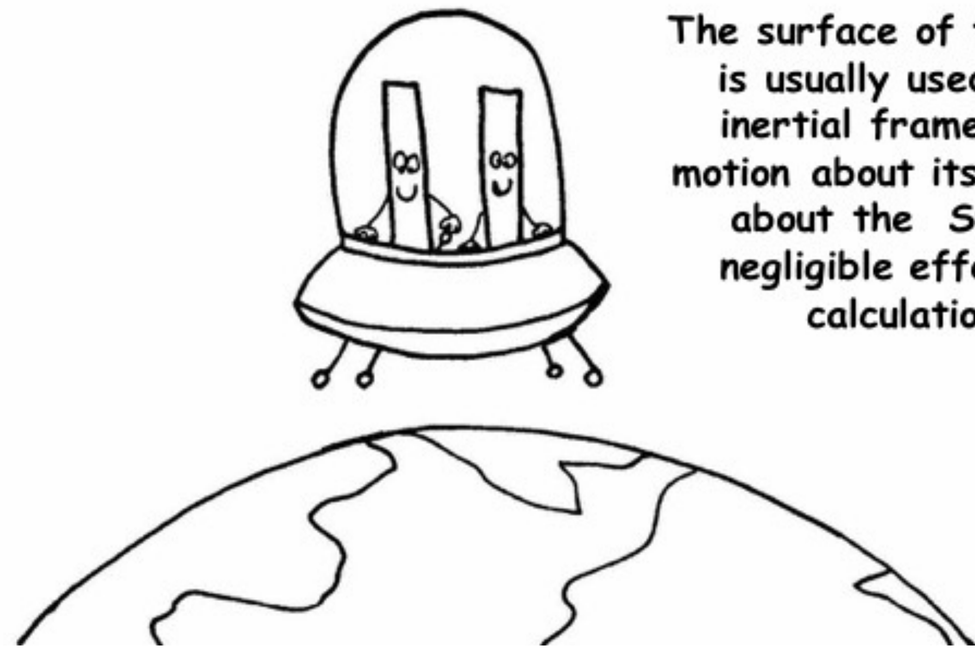


The same situation would happen if Bud turns the car while maintaining his speed. The suitcase will then start to move in a direction that is opposite to the turn. According to Bud (who is in an accelerated frame), Newton's first law is invalid, while for Maggie (who is in an inertial frame), it is valid because to her when Bud turns, the suitcase tends to continue its uniform motion along a straight line and therefore moves in a direction opposite to the turn.

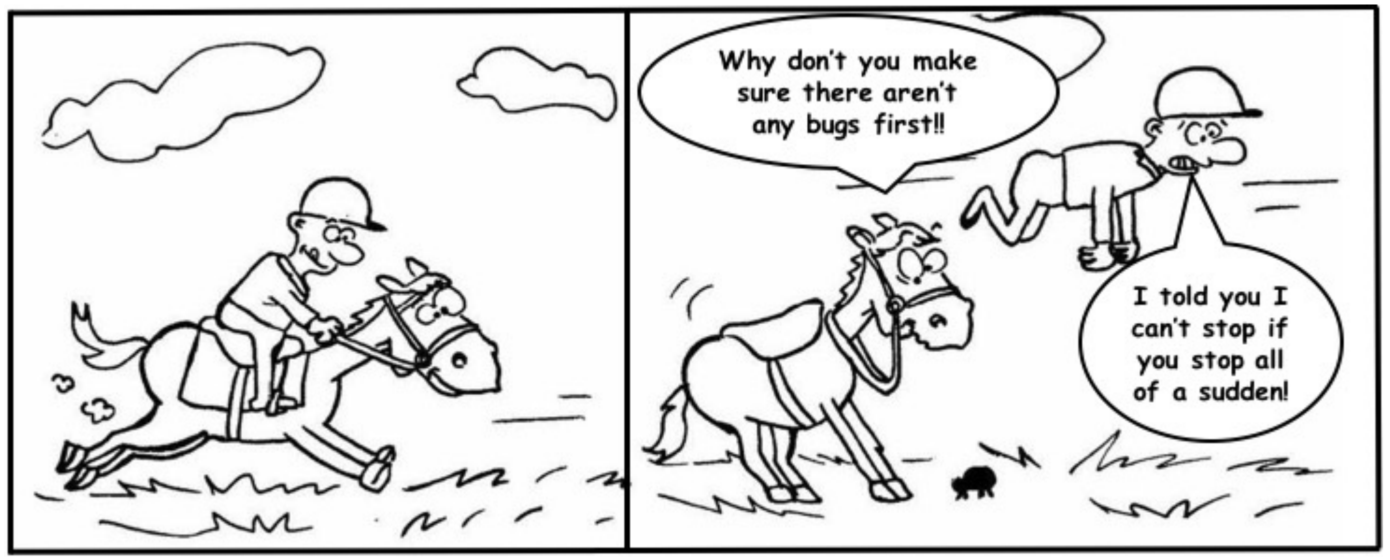




Any reference frame moving at a constant velocity relative to an inertial frame is also an inertial frame.



The surface of the Earth is usually used as an inertial frame as its motion about its axis and about the Sun has negligible effects on calculations



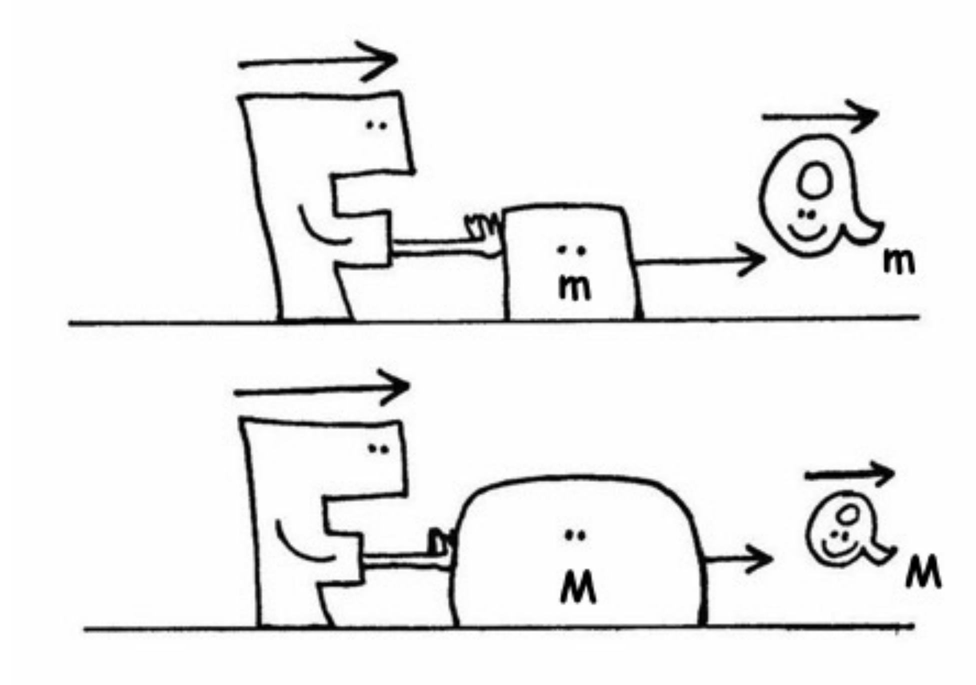
Despite that some quantities have different values in different inertial frames, the laws of physics are the same in all inertial frames of reference. This follows from the fact that there is no absolute universal frame of reference. If the laws of physics were different in different inertial frames, i.e. for different observers in relative motion, then they would be able to distinguish from these differences which of them is stationary in space and which is moving, setting up an absolute reference frame which does not exist.

Newton's Second Law:

We have seen that Newton's first law is concerned with the situation when the net force acting on an object is zero (either there is no force or the resultant of existing forces is zero). Newton's second law explains what happens if there is one or more forces acting on an object and the net force is not zero.

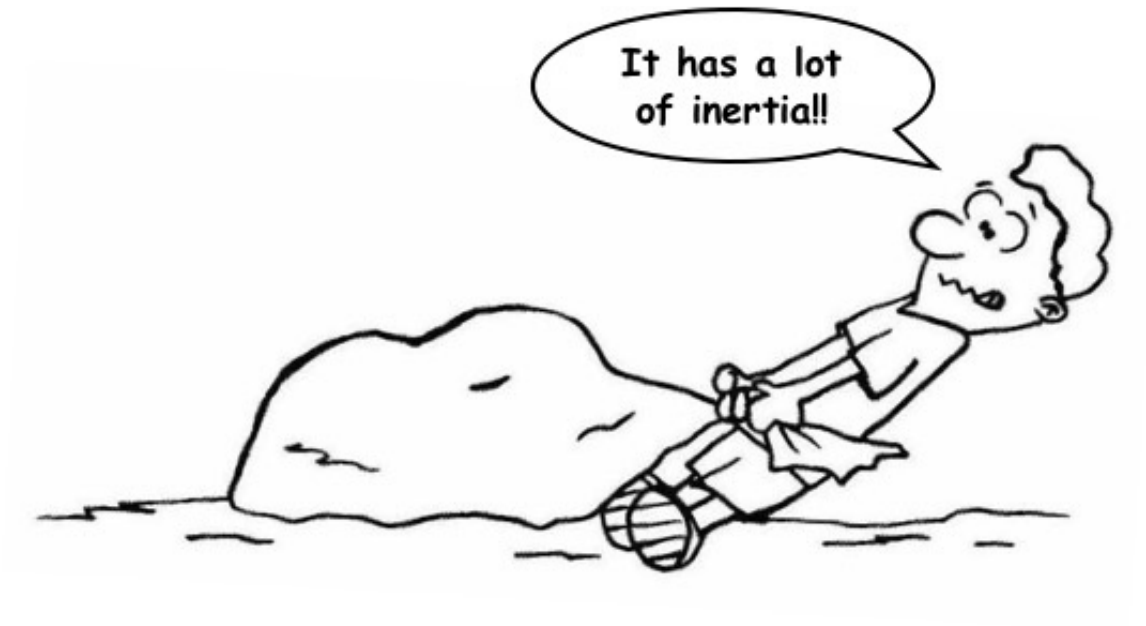
Suppose the same force is applied to two different masses, the larger mass M will have a lower acceleration such that

$$\frac{m}{M} = \frac{a_M}{a_m}$$



$$M > m$$
$$a_M < a_m$$

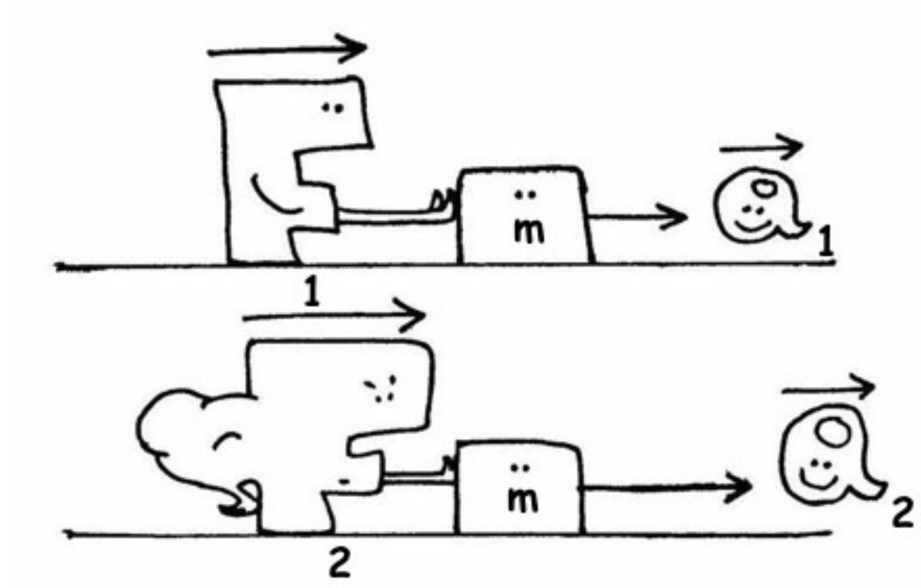
Therefore, the more massive the object is, the harder it is to accelerate it, and so mass is the property of an object associated with how much it resists changes in its velocity. As discussed earlier, this resistance is known as Inertia, and therefore mass is a measure of inertia where it relates the acceleration of an object to the force acting on it.



Mass is an inherent characteristic of matter independent of the way of measurement or the surroundings.

It is found experimentally that when two masses m_1 and m_2 are combined together, the resultant body behaves as a mass of $m_1 + m_2$. Mass is thus a scalar quantity obeying the rules of ordinary arithmetic.

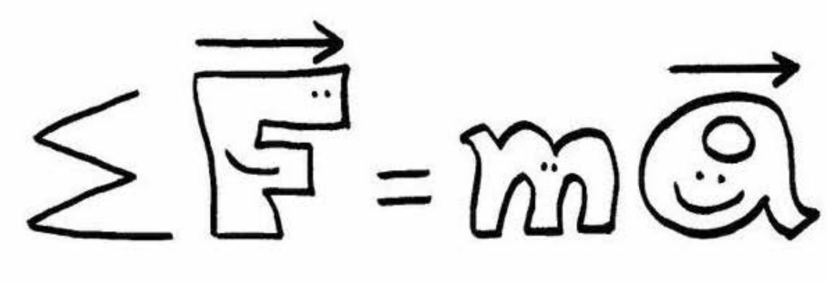
Now suppose that the same mass is acted upon by two different forces, one is small and the other is a large bodybuilder. As can be seen below, the larger force will generate a larger acceleration



$F_2 > F_1$

$a_2 > a_1$

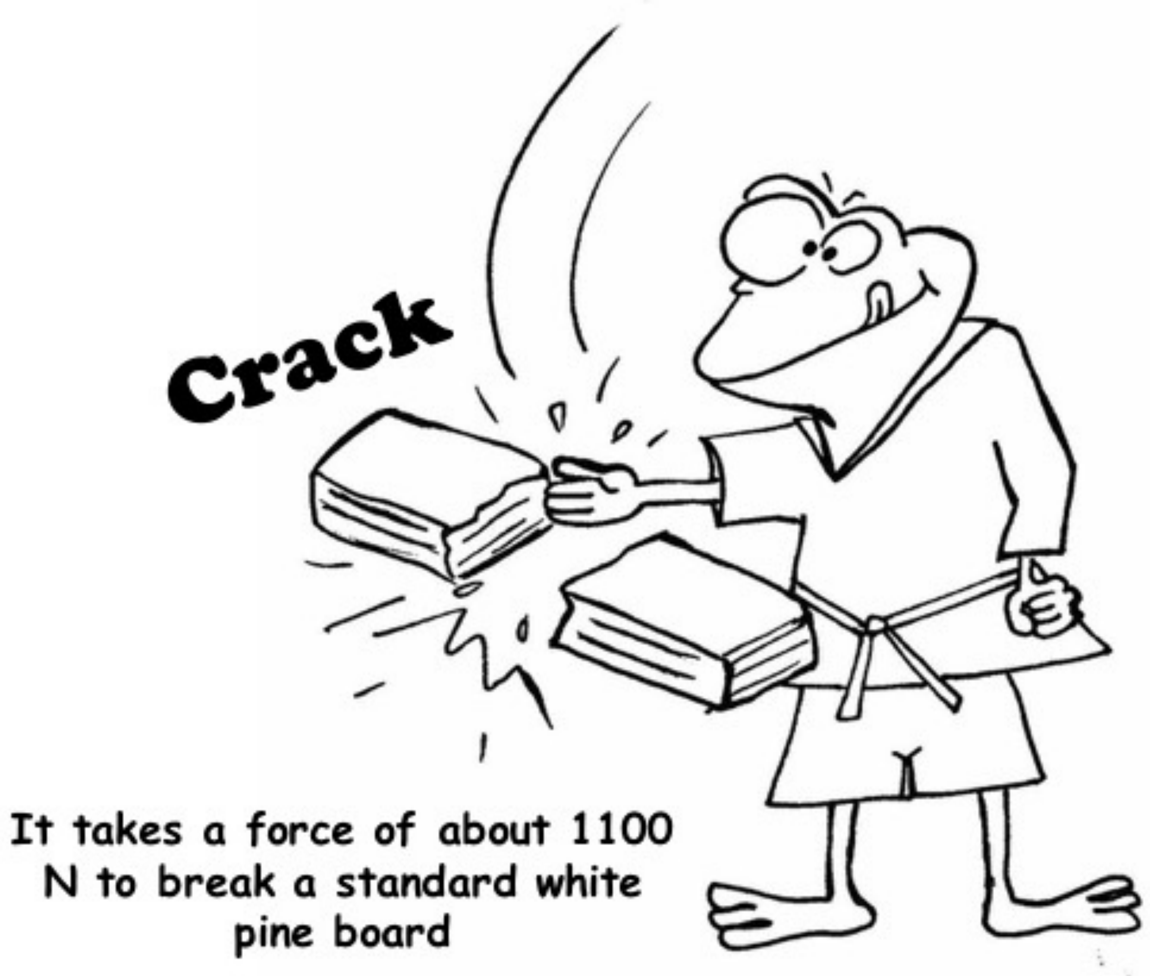
This is summarized in Newton's second law: *The acceleration of an object is directly proportional to the net force acting on it, in a direction parallel to it, and inversely proportional to its mass*



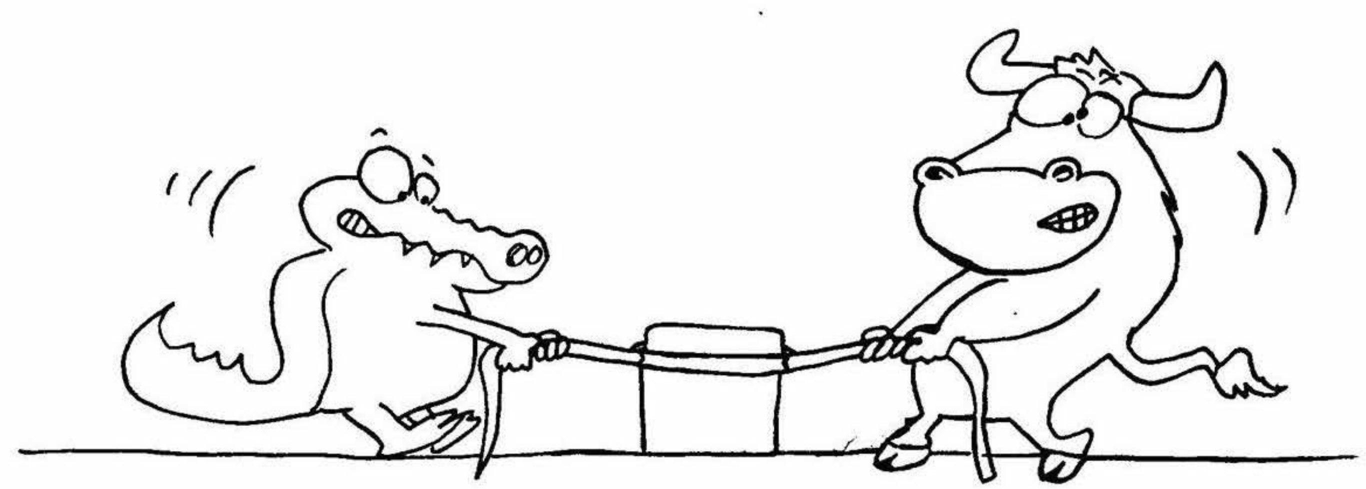
The unit of force is the Newton: 1 Newton is the force needed to accelerate 1 kg of mass at a rate of 1 m/s^2

$$1\text{N} = 1\text{kg}.\text{m/s}^2$$

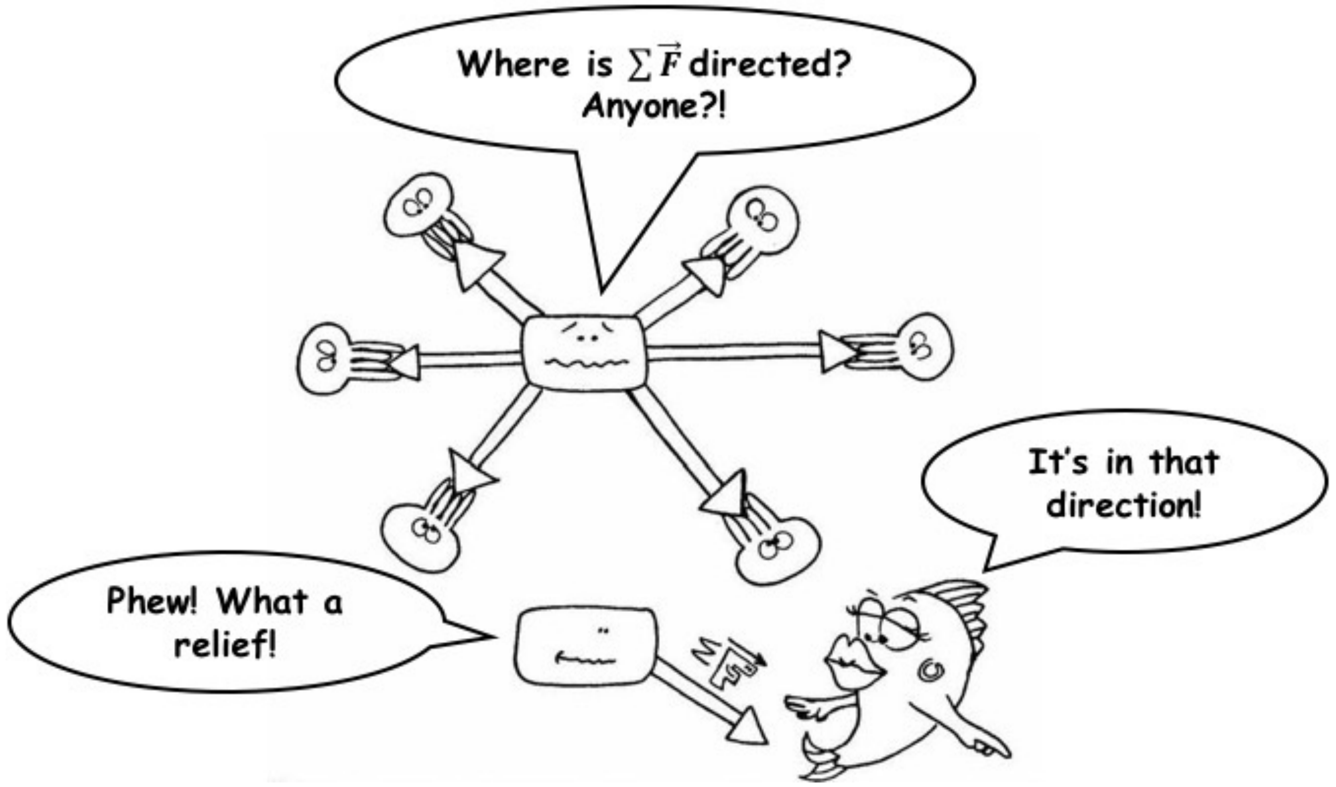
Since force is a vector quantity, the resultant force $\sum\vec{F}$ is the vector sum of all of the forces.



Newton's second law shows that if $\sum\vec{F} = 0$, then $\vec{a} = 0$ and the object will either remain at rest or continue to move with a constant velocity (constant speed in a straight line). This is what happened when a bull and an alligator decided to test who is stronger. All other animals where watching and cheering them up. Surprisingly, the block remained stationary which means they both are pulling with the same magnitude of force and since they are pulling in the exact opposite directions, the net external force on the block is zero and it will not move.



If $\sum\vec{F} \neq 0$, then the object will accelerate in the direction of the net force ($\sum\vec{F}$) like the case of this block that was pulled by six squids for reasons yet to be known.



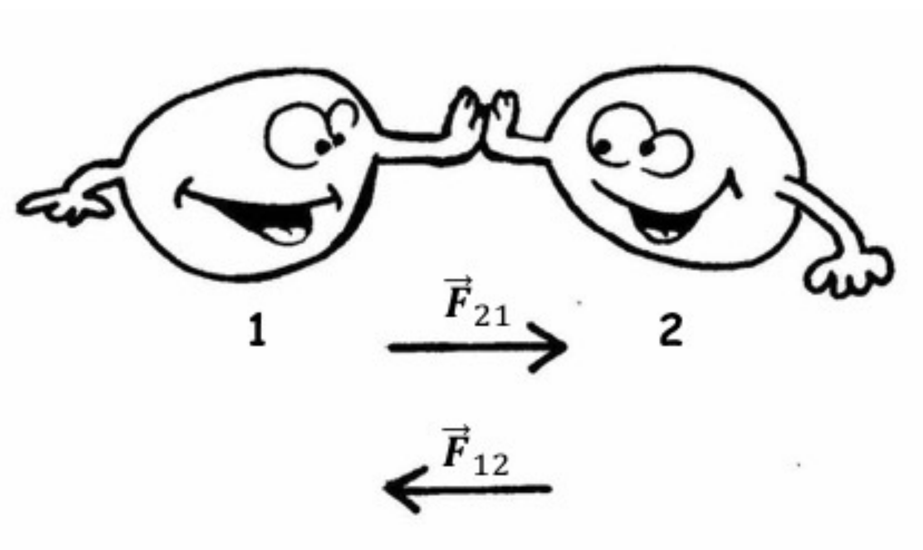
.

Newton's Third Law:

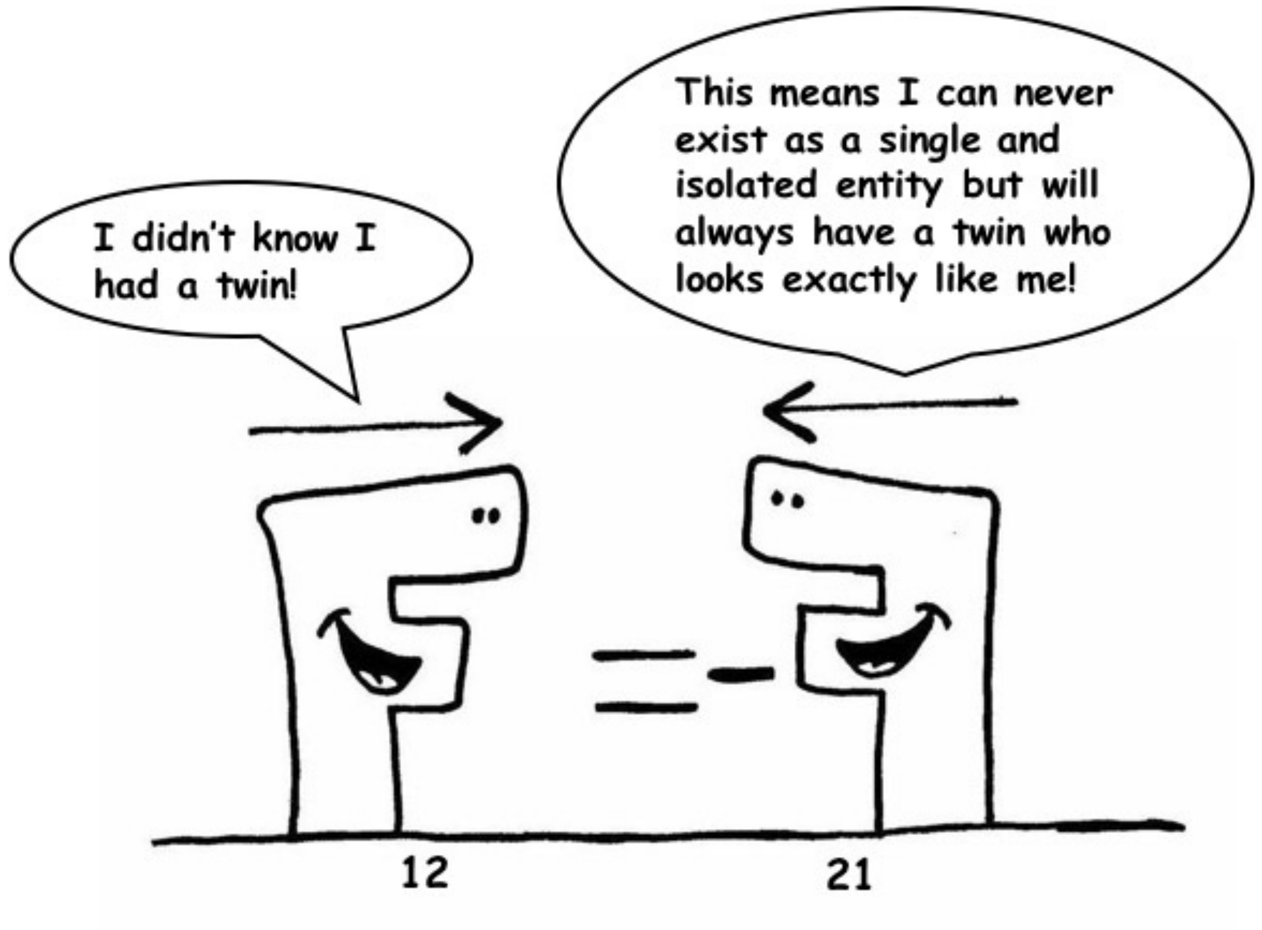
Newton's third law shows the fact that a force acting on an object is always due to its interaction with another object. Let's say you kick a ball, then your foot will exert a force (action) on the ball, and the ball will in turn exert a reaction force on your foot. The heavier the ball, the more painful that reaction force would feel.

Therefore, forces always come in pairs and a single isolated force does not exist. In other words, if body 1 exerts a force $\vec{F}_{21}$ on body 2, then body 2 will exert an equal and opposite force $\vec{F}_{12}$ on body 1

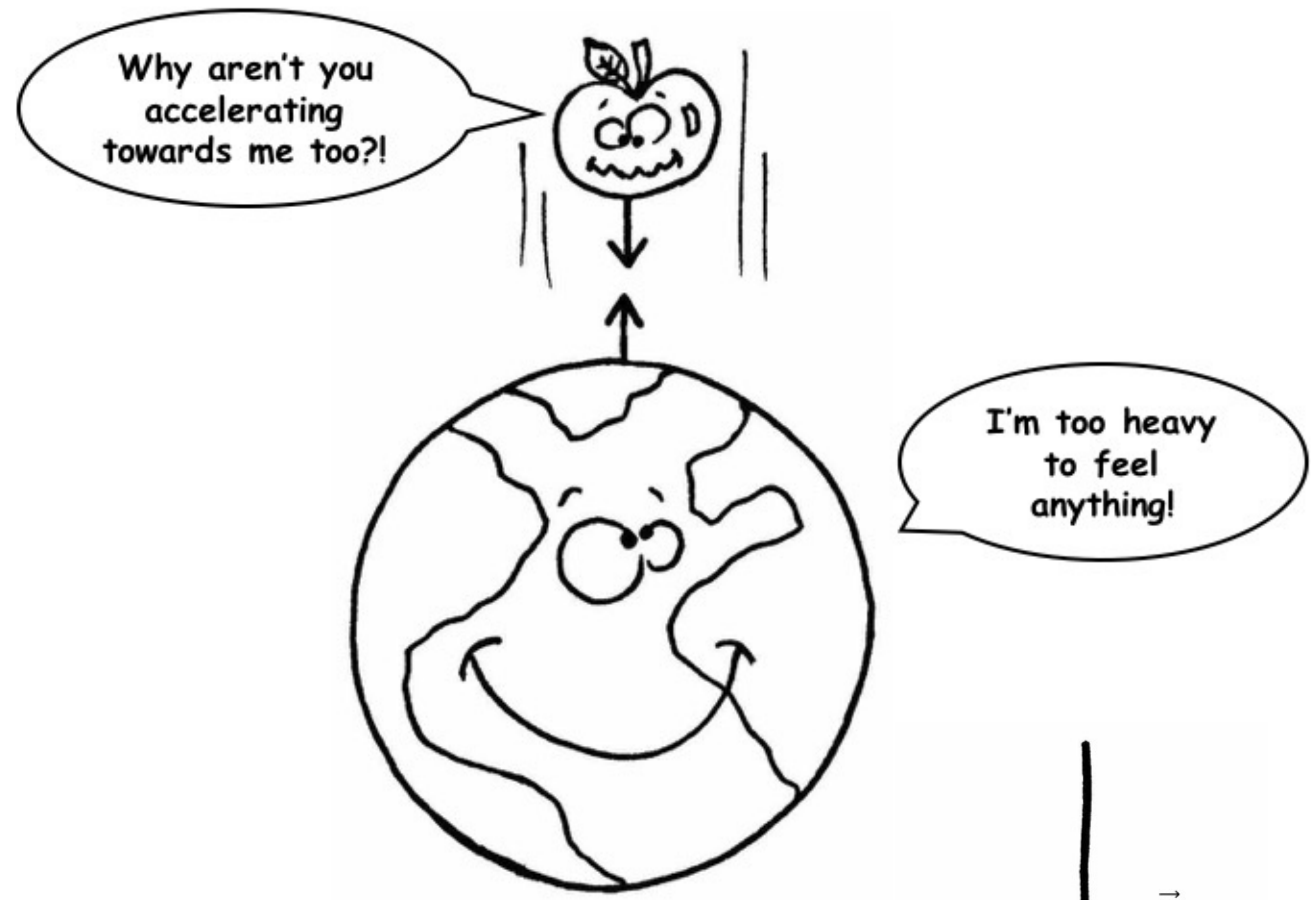
$$\vec{F}_{21} = -\vec{F}_{12}$$



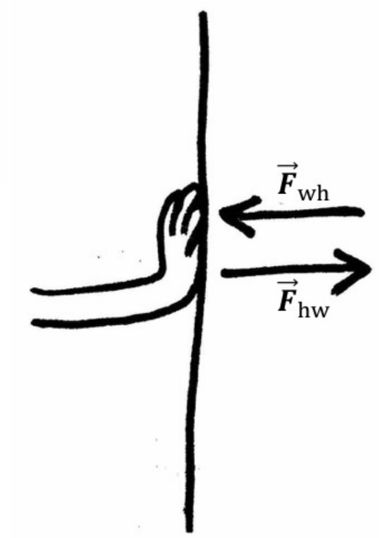
Newton's third law can sometimes be stated as *to every action there is an equal and opposite reaction*. Note that we can consider either force as an action and the other as a reaction, i.e. it does not imply a cause and effect relationship. In addition, action and reaction forces always act on different bodies so they can't cancel each other out.

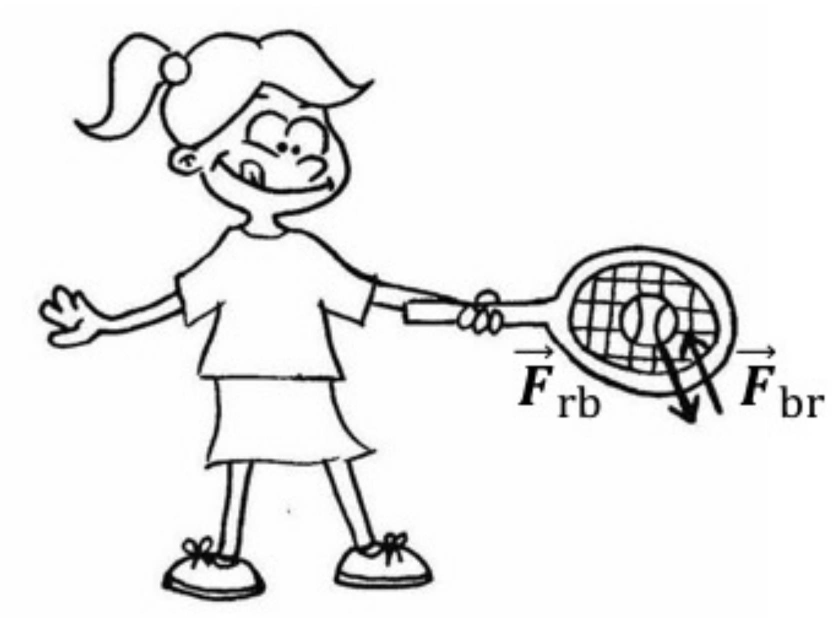


Both the apple and the Earth exert equal and opposite forces ($\vec{F}_{aE} = -\vec{F}_{Ea}$) on each other, but because the mass of the Earth is very large (M_E = 5.972 x10^{24} kg), its acceleration $a_E = F_{Ea}/M_E \approx 0$ is close to zero.



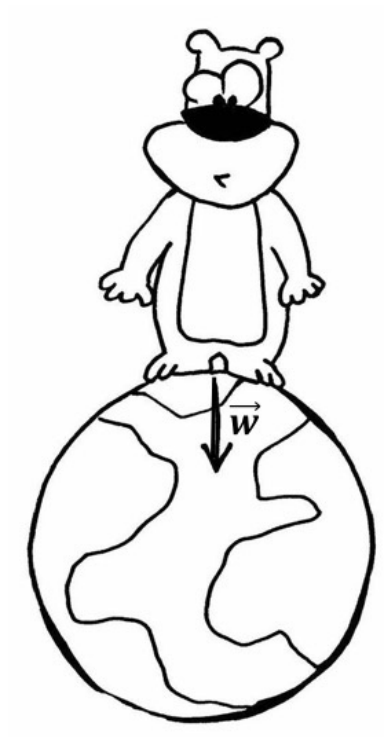
If you push the wall, the wall pushes back with an equal and opposite force. The same goes for a tennis racket hitting a ball.





Some Particular Forces:

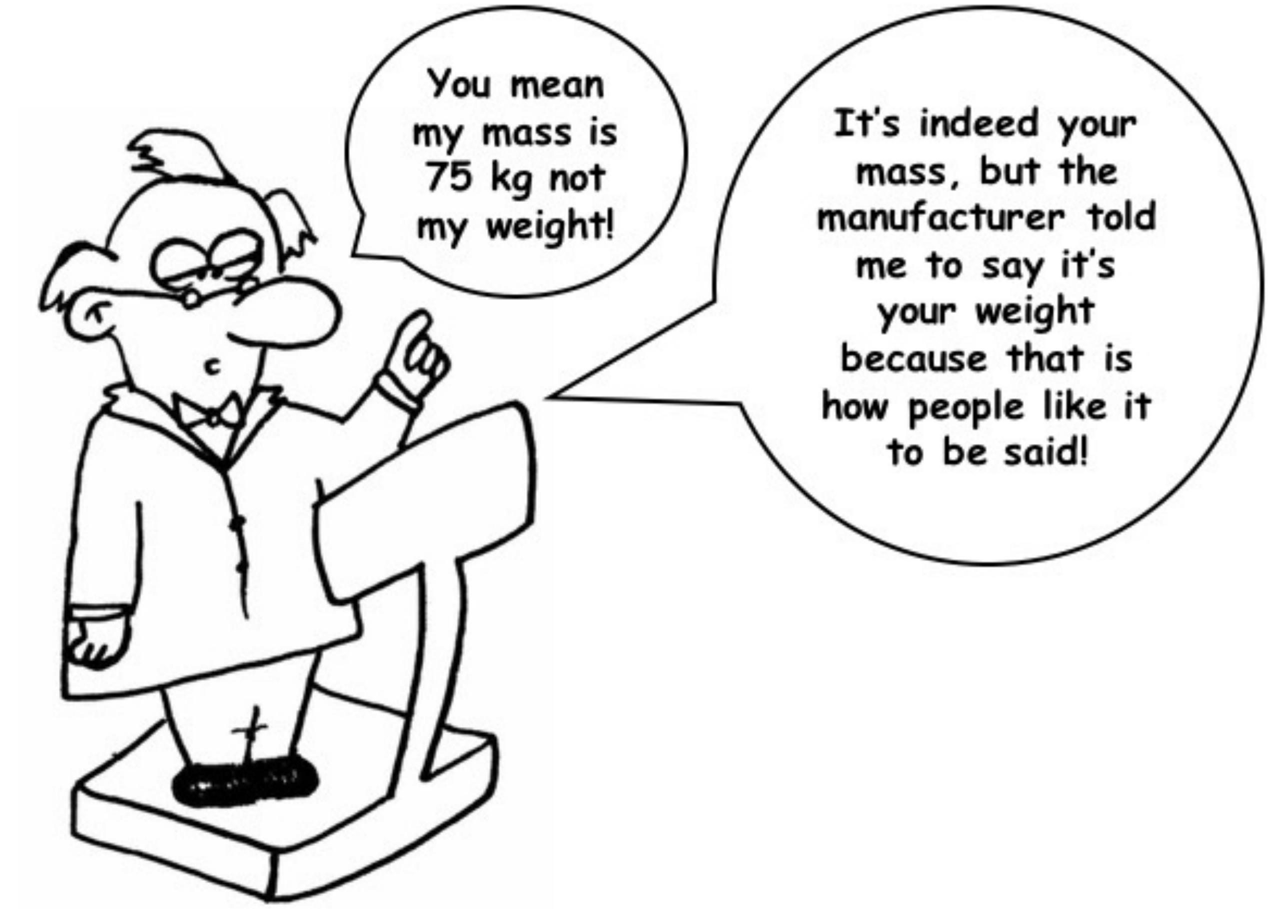
Weight $\vec{w}$:



Weight is the gravitational force exerted by an astronomical body (such as the Earth) on an object that is relatively smaller in mass and size. Near the Earth's surface, the gravitational acceleration is g = 9.8 m/s^2. Using Newton's second law, we find that

$$\vec{w} = m\vec{g}$$

One very common way of measuring mass is by using the weight of an object. The mass is equal to the weight (the gravitational force) divided by the gravitational acceleration g. The term weight is often misused to indicate the mass in conversations, but as you have seen, the two are different quantities, mass is in kg and weight is a force and has the unit of Newton. Therefore, the weighing scale divides the weight by g and displays to us the mass in kg.

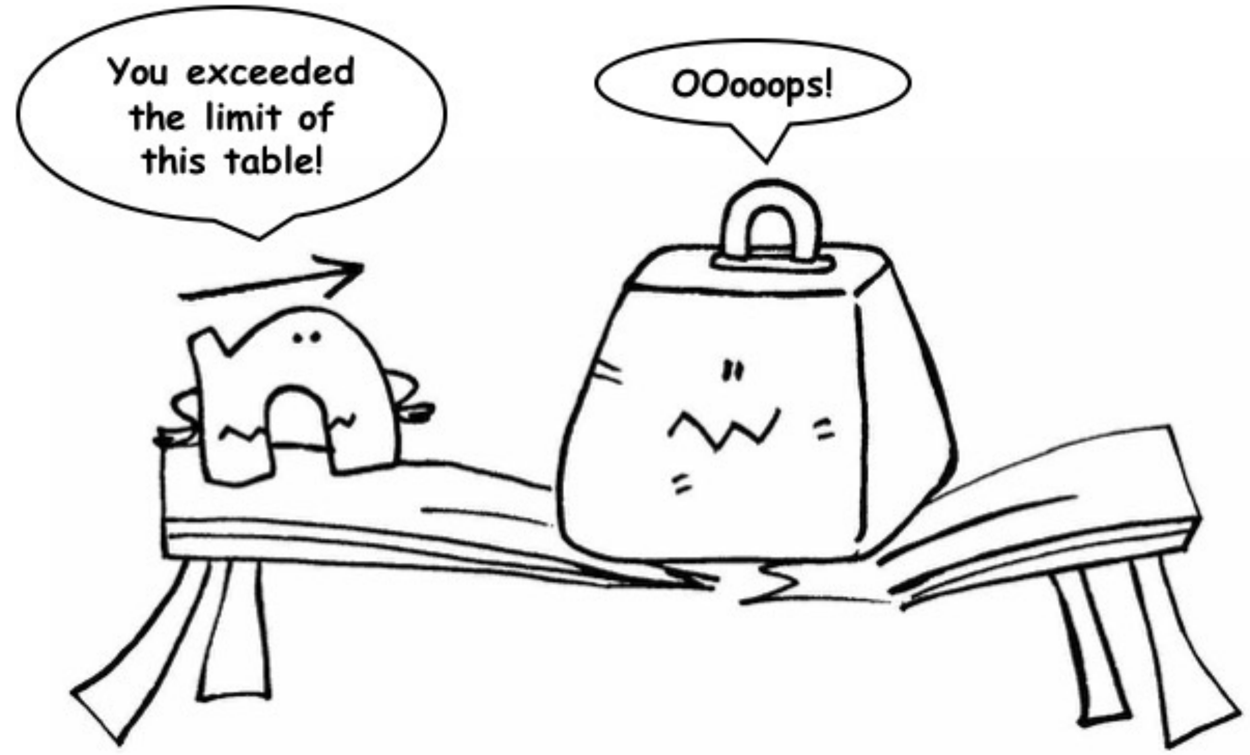


Note that the mass in the equation of the weight determines the gravitational strength between an object and the Earth, and is known as the gravitational mass. While the mass we defined earlier is a measure of inertia (the resistance to the change in the state of motion) and is known as inertial mass. However, repeated experimental measurements have shown that inertial and gravitational mass have the same value.

Because the weight $\vec{w}$ of an object depends on g, which decreases with altitude (as we shall see later), weight is not an inherent property of matter. You may think that you lost weight (i.e. mass) if you weigh yourself on a mountain or inside an airplane, but the reality is that your mass is all the same, it is just the gravitational pull that is less at higher altitudes.

The Normal Force: 

When an object is in contact with a surface, whether at rest or moving on it, the surface will exert a force on the object that is always perpendicular (normal) to the surface of contact. This force is known as the normal force $\vec{n}$ and

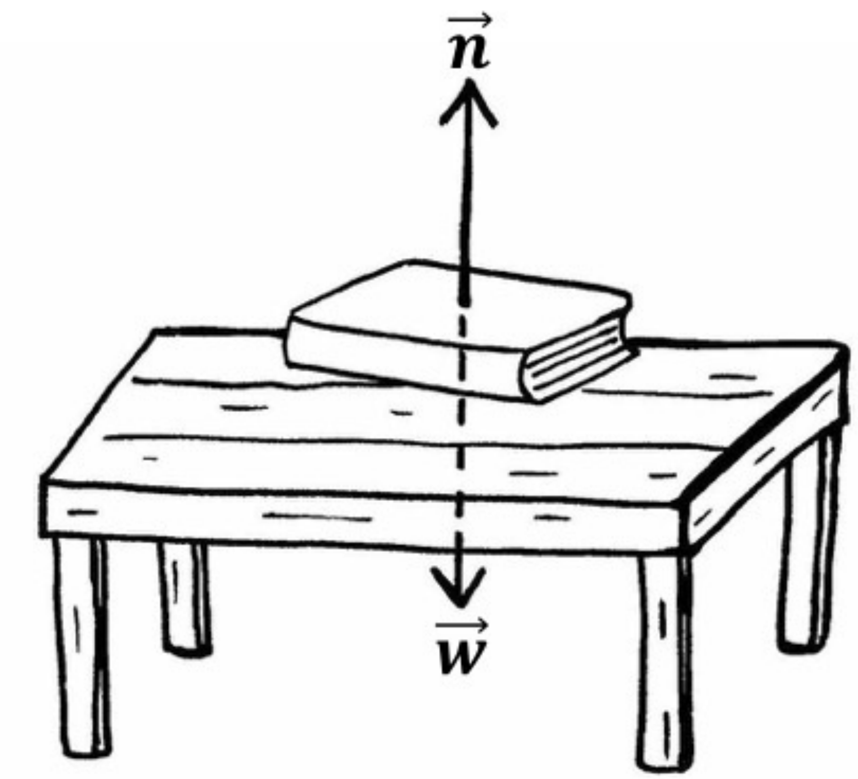


is what prevents the object from falling through the surface and it can have any value up to the point where the surface can't take it anymore and breaks.

Consider the book shown below resting on a table. Since the book is at rest, its acceleration is zero and from Newton's second law it follows that the net force acting on the book is zero

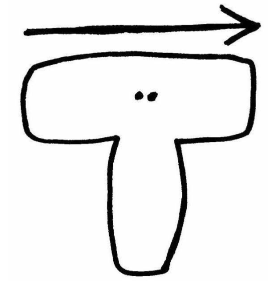
$$\sum \vec{F} = \vec{n} - \vec{w} = 0$$

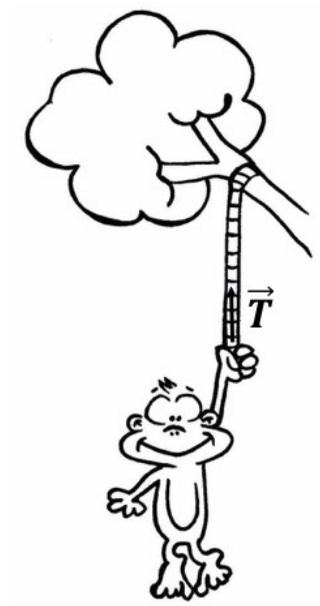
$$\Rightarrow \vec{n} = \vec{w} = m\vec{g}$$



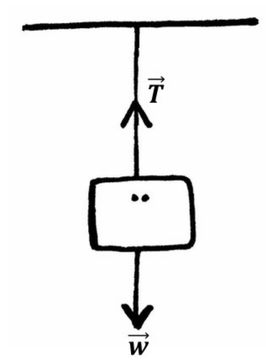
Therefore, in this case, the weight of the book is balanced by the normal force. Note that the weight and the normal force are not an action and reaction pair (action and reaction forces always act on different bodies so they can't cancel each other out). The reaction to the weight $\vec{w}$ is the force that the book exerts on the Earth. While the reaction to the normal force $\vec{n}$ is the force exerted by the book on the table.

The Tension Force:





When an object is attached to a rope, the rope exerts a force on the object $\vec{T}$ that is directed along the rope away from the object at the point of contact between them. In solving problems, the mass of the rope is usually ignored (referred to as a light rope) and it is assumed to be non-stretchable. In this case the magnitude of $\vec{T}$ is the same at all points in the rope.



Consider a block that is hanging to the ceiling by a light rope. Since it is at rest, its acceleration is zero and by applying Newton's second law in the vertical direction we have

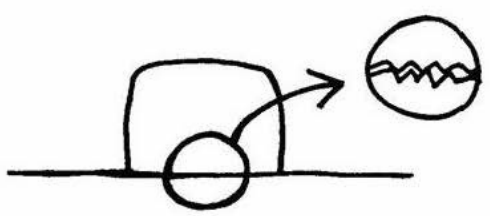
$$\sum \vec{F} = \vec{T} - \vec{w} = 0$$

$$\Rightarrow \vec{T} = \vec{w} = m\vec{g}$$

Friction:

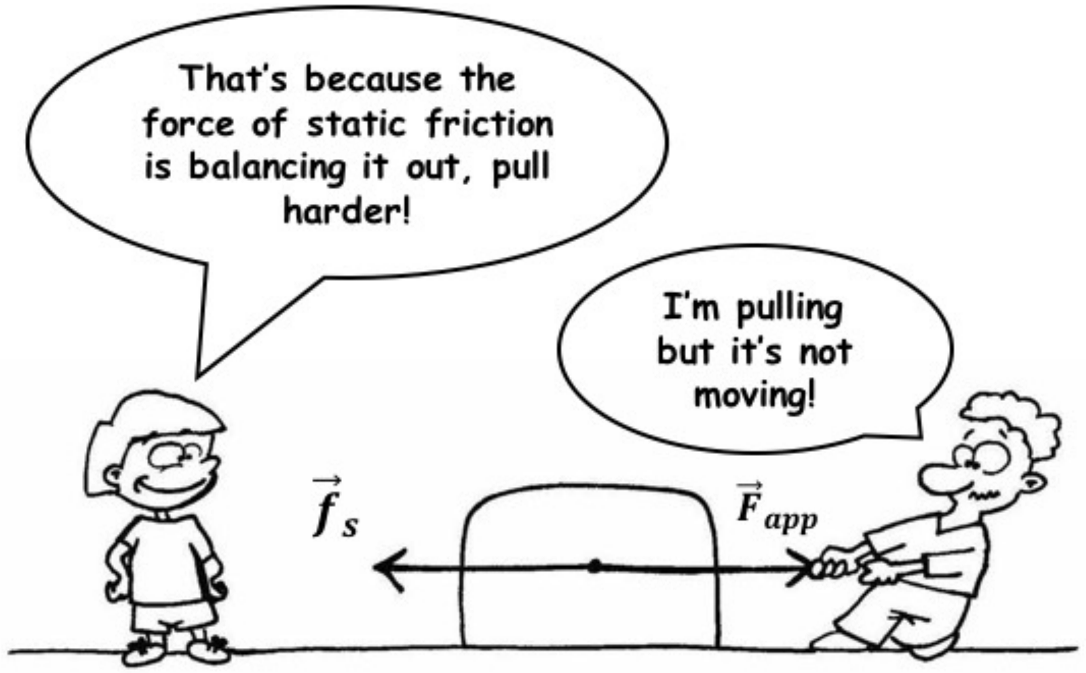
Friction plays a very important role in our everyday life. Imagine living in frictionless land, simple activities such as walking, holding an object or driving a car would be impossible to do.

Friction is a complex phenomenon, resulting from the interaction between surface atoms or molecules of any two objects in contact (these forces are fundamentally electrostatic in nature). The actual area of contact at the microscopic level is much smaller than what appears at the macroscopic level as shown, so one peak would physically obstruct the other. In addition, chemical bonds would continuously form and break between the two surfaces. As a result, extremely smooth surfaces can actually increase friction due to the increased interactions between the molecules of the surfaces, this is why a lubricating oil film reduces friction by preventing the surfaces from coming into contact with each other.

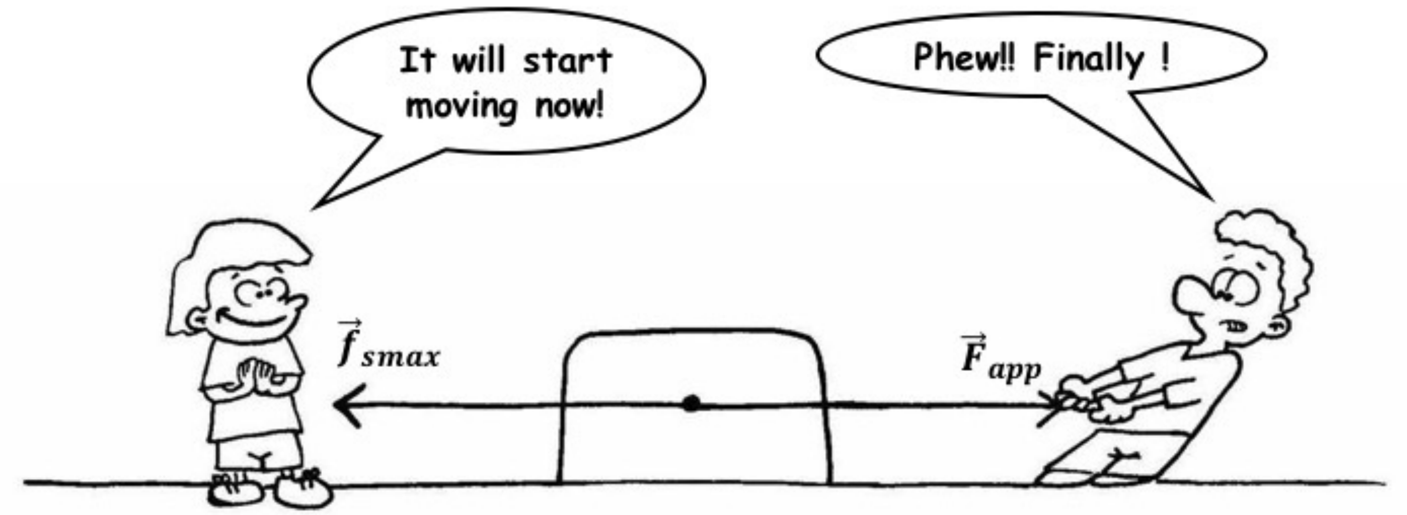


The direction of the frictional force is always parallel to the surface of contact opposite to the relative motion between the two surfaces (or opposite to the component of the applied force parallel to the surface if the object is at rest). Hence, both the frictional force and the normal force are contact forces and they are always perpendicular to each other.

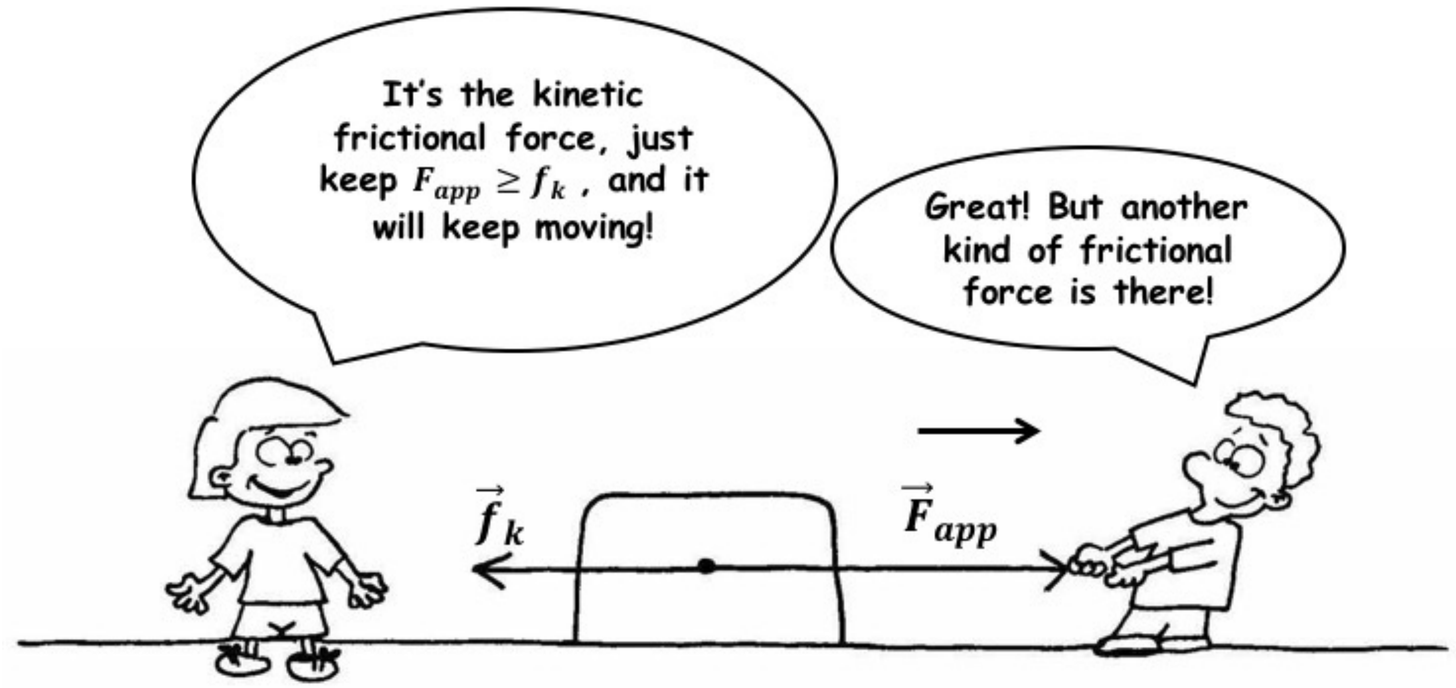
Suppose that Eric is trying to move a box. Even though he is applying a force $\vec{F}_{app}$ to the left, it is being balanced by the statistical frictional force $\vec{f}_s$ to the right and so, much to Eric's frustration, it remains stationary. Sophie is cheering him up to keep pulling.



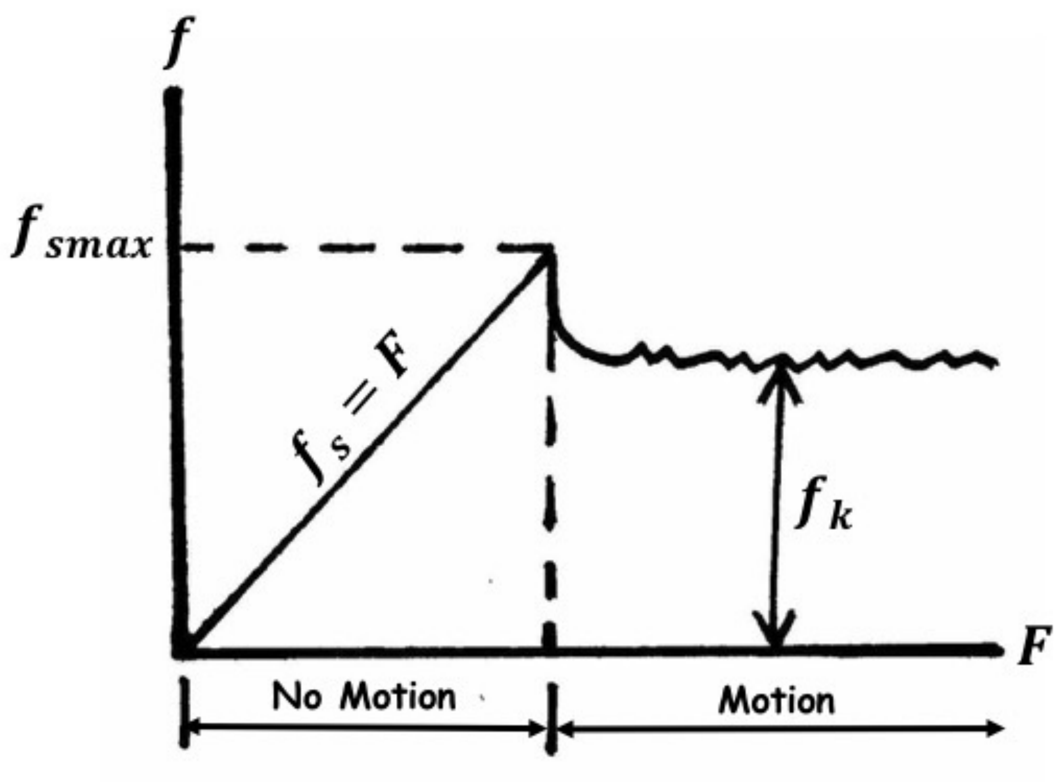
When Eric increases the applied force, the statistical frictional force increases by the same amount and the box will still remain at rest. When the applied force reaches the maximum statistical frictional force possible $\vec{f}_{smax}$, then the box will start to accelerate to the left.



As the box moves, the kinetic frictional force $\vec{f}_k$ will then act on the box and its value is less than $\vec{f}_{smax}$.



The graph below shows how the frictional force changes with the applied force

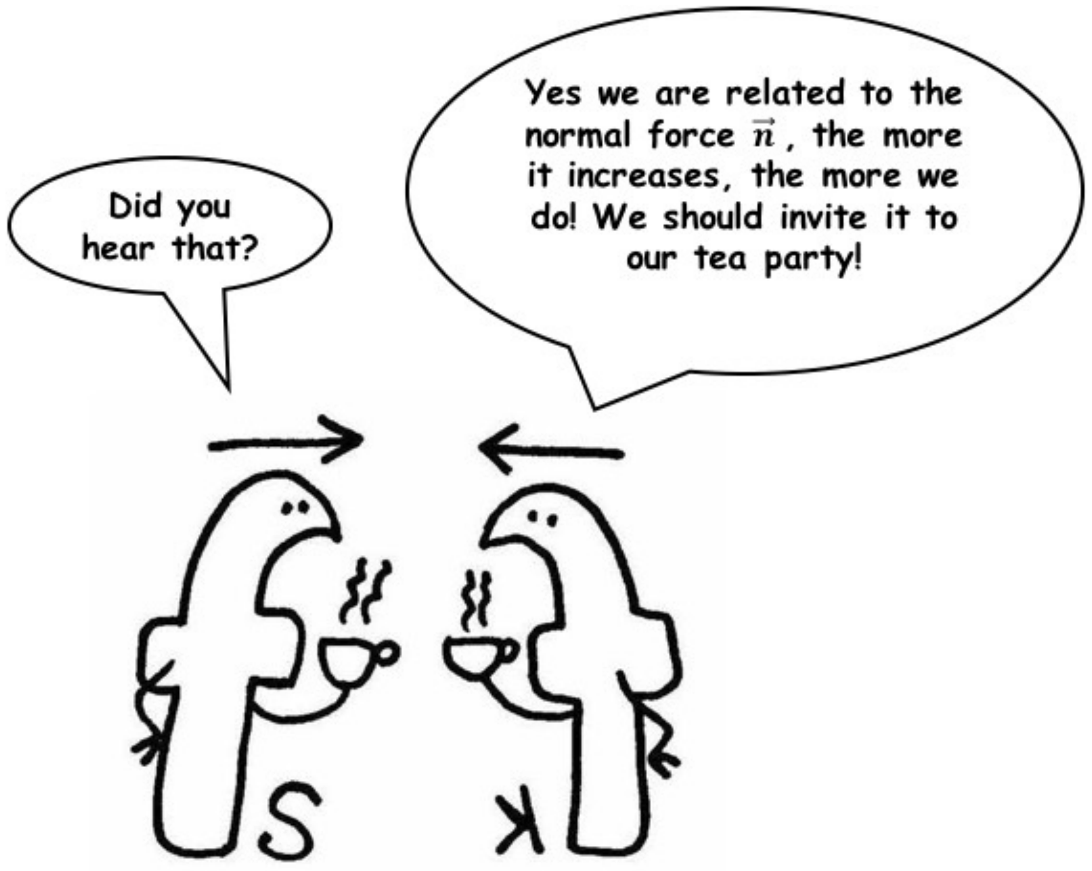


If $F_{app} < f_k$ or $F_{app} = 0$ (Eric stops pulling), then the box will decelerate until it comes to rest. If $F_{app} = f_k$, then the box will move at a constant speed and if $F_{app} > f_k$, the box will accelerate. Experimentally it is found that the magnitudes of both $\vec{f}_s$ and $\vec{f}_k$ are related to the normal force $\vec{n}$ through the relations

$$f_s \leq \mu_s n$$

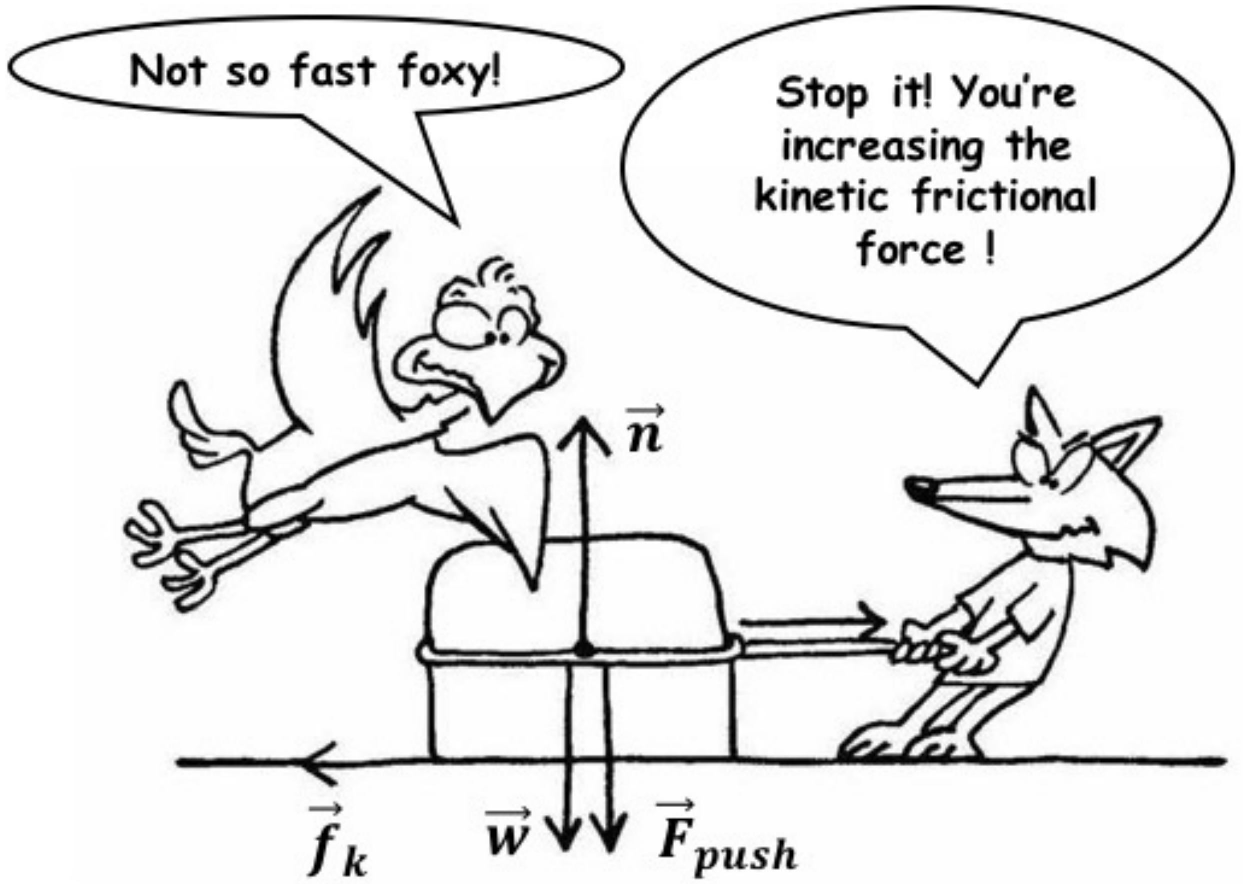
$$f_{smax} = \mu_s n$$

$$f_k = \mu_k n$$

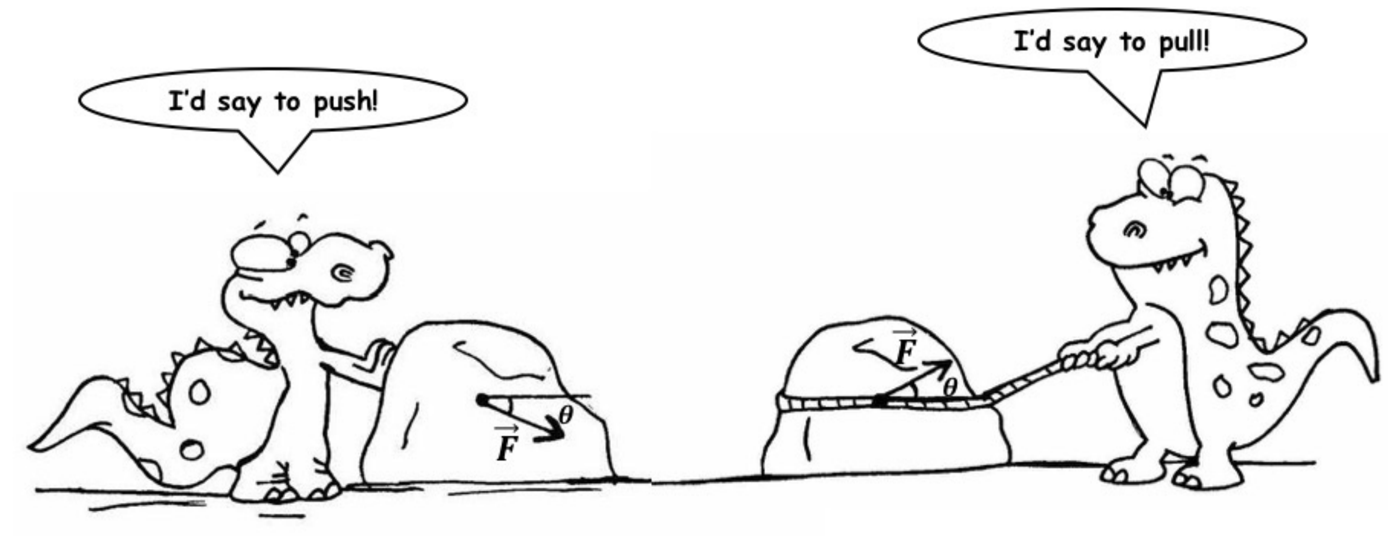


where μ_s and μ_k are the coefficients of the static and kinetic friction respectively. As an example, for wood on wood, μ_s has a value ranging between 0.25-0.5 and μ_k has the value of 0.2.

Now consider our friend bird who discovered that the fox was stealing something from his pantry, as a first line of defense, he increased the normal force on the container by pushing it down resulting in $n = w + F_{push} > w$, which in turn will increase both the statistical and kinetic frictional forces.



One question we want to ask is if it is easier to pull or push. It seems the Dinosaurs in the lost world where asking the same question too!



If the Dino pushes the rock at some angle, then the normal force will increase

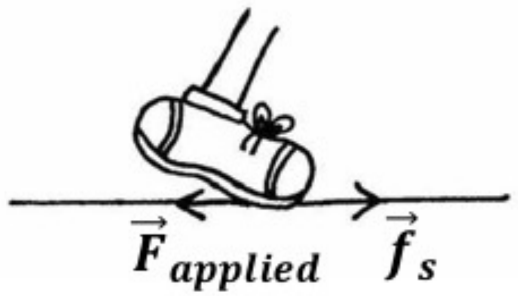
$$n = mg + Fsin\theta > mg$$

where $Fsin\theta$ is the vertical component of the applied force. This in turn will increase the force of kinetic friction ($f_k = \mu_k n$). While if the other Dino pulls the rock at some angle, then the normal force will decrease

$$n = mg - Fsin\theta < mg$$

which in turn will decrease the force of kinetic friction. Therefore, it takes less effort to pull than to push.

Remember what it was like in frictionless land? Now let's see how walking is possible in the normal world



When the foot applies a force on the ground, the horizontal component of which is $\vec{F}_{app}$ to the right, the statistical force then acts to the left and balances it out which prevents the foot from slipping.

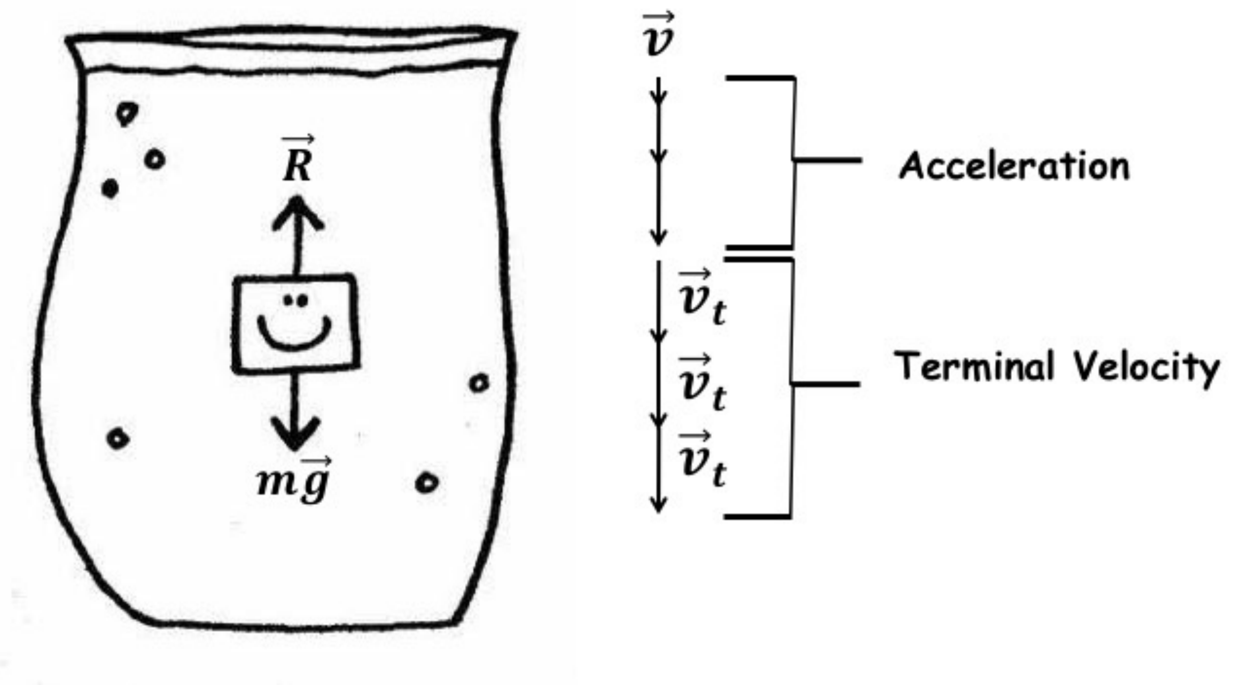
Fluid Resistance:

Another type of frictional force is the resistive force $\vec{R}$ experienced by an object in relative motion to a fluid (such as a gas or liquid). The direction of $\vec{R}$ is always opposite to the direction of the object's velocity relative to the fluid. You can feel this force if you put your hand out of the window of a fast-moving car.

The magnitude of $\vec{R}$ generally increases with the object's speed but can have a complex dependence on speed. We will consider two cases. The first is for an object falling through a liquid with a low speed. A simplified effective model is to assume that the resistive force is approximately proportional to the object's velocity as

$$\vec{R} = -b\vec{v}$$

Where $\vec{v}$ is the velocity of the object relative to the fluid, and b is a constant that depends on the properties of the fluid and on the shape and size of the object. The negative sign indicates that $\vec{R}$ is opposite to $\vec{v}$.

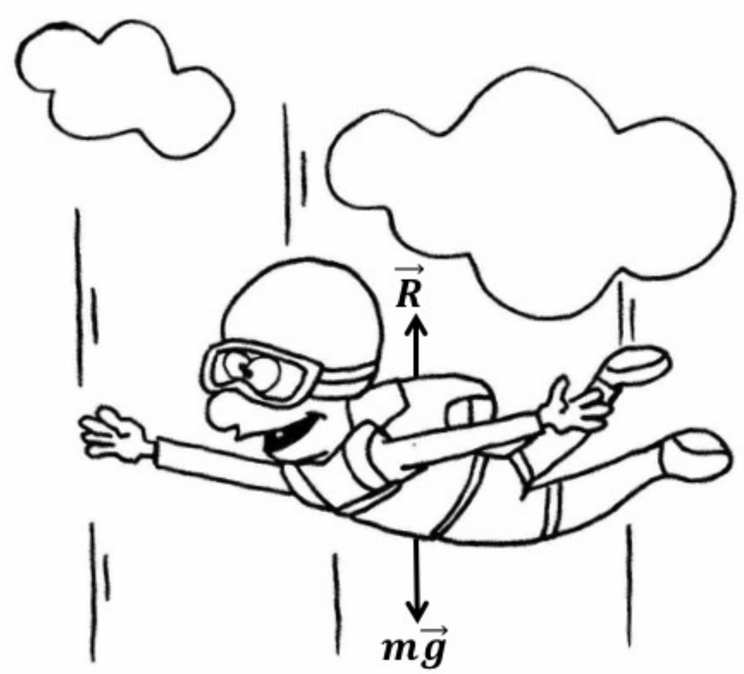


As the object falls, the resistive force will continuously increase since the object's speed is increasing due to Earth's gravity. The resistive force will then reach a point where it becomes equal to the weight of the object, and by applying Newton's second law, the object will then have zero acceleration and it will fall at a constant speed known as the terminal speed v_t. A simple calculation shows that

$$v_t = mg/b$$

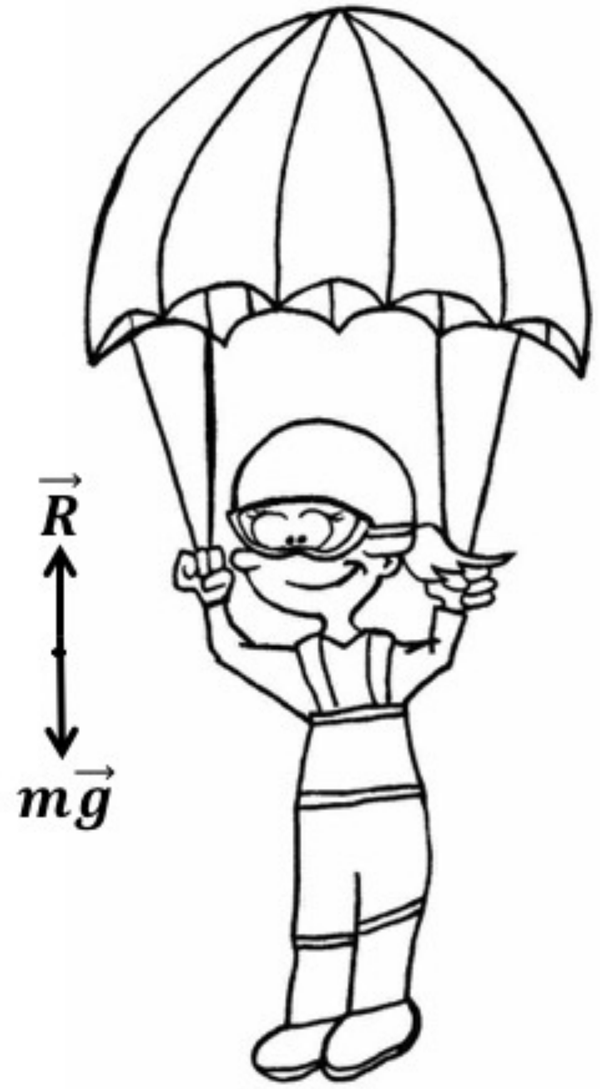
The second case is for a large object moving at high speed through air such as a skydiver, a baseball, a raindrop or a car moving at high speed. The resistive force (known as air drag) can then be approximated to be proportional to the square of the speed as

$$R = \frac{1}{2} c\rho A v^2$$



Where ρ is the density of air, A is the effective cross-sectional area and c is the drag coefficient. There is also a small upward buoyant force that we neglect here. As in the first case, the resistive force increases continuously as the skydiver falls, until it becomes equal to his weight. His acceleration will then become zero and he would fall with the constant terminal speed v_t given by

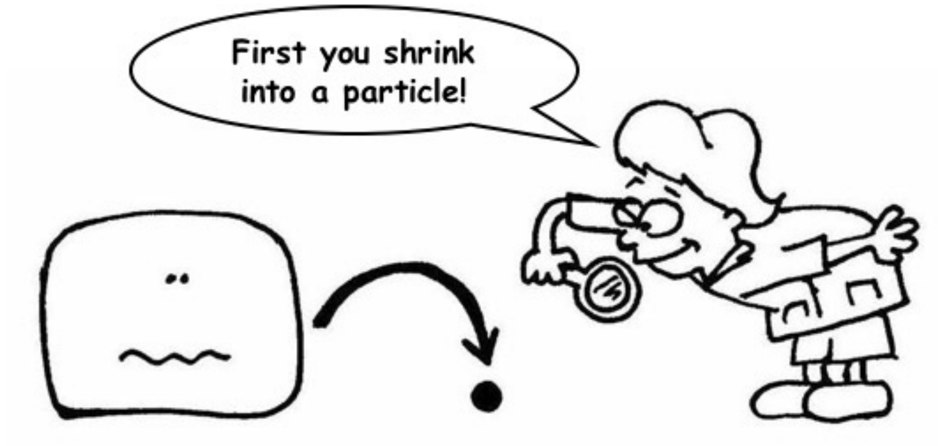
$$v_t = \sqrt{\frac{2mg}{c\rho A}}$$

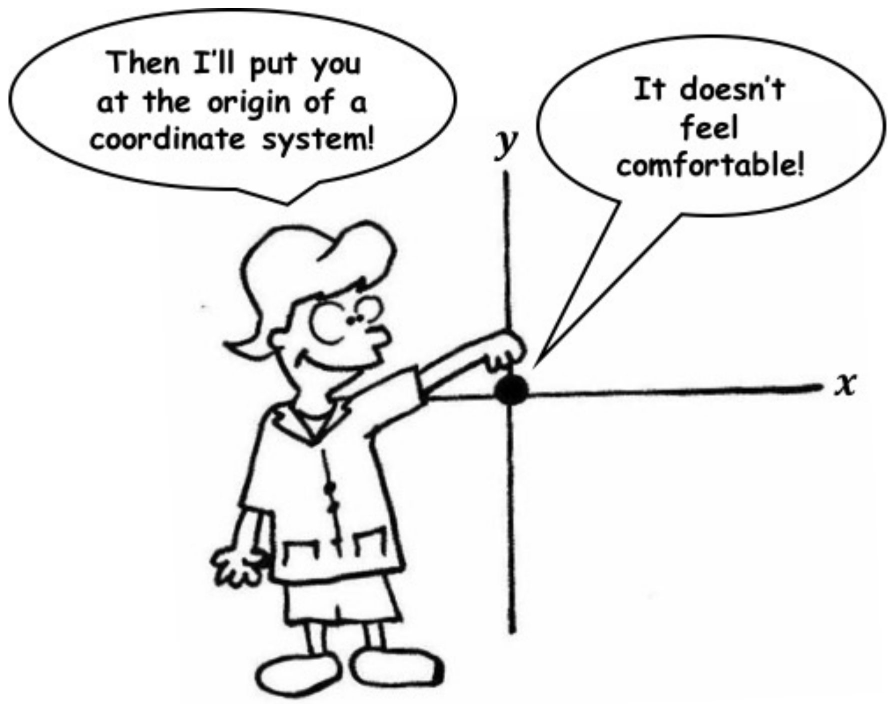


Therefore, to adjust the terminal speed of the fall, a skydiver can change the positions of the arms and legs which in turn will change the effective cross-sectional area A. To float safely to the ground, a skydiver increases air resistance by using a parachute, which significantly decreases the terminal velocity.

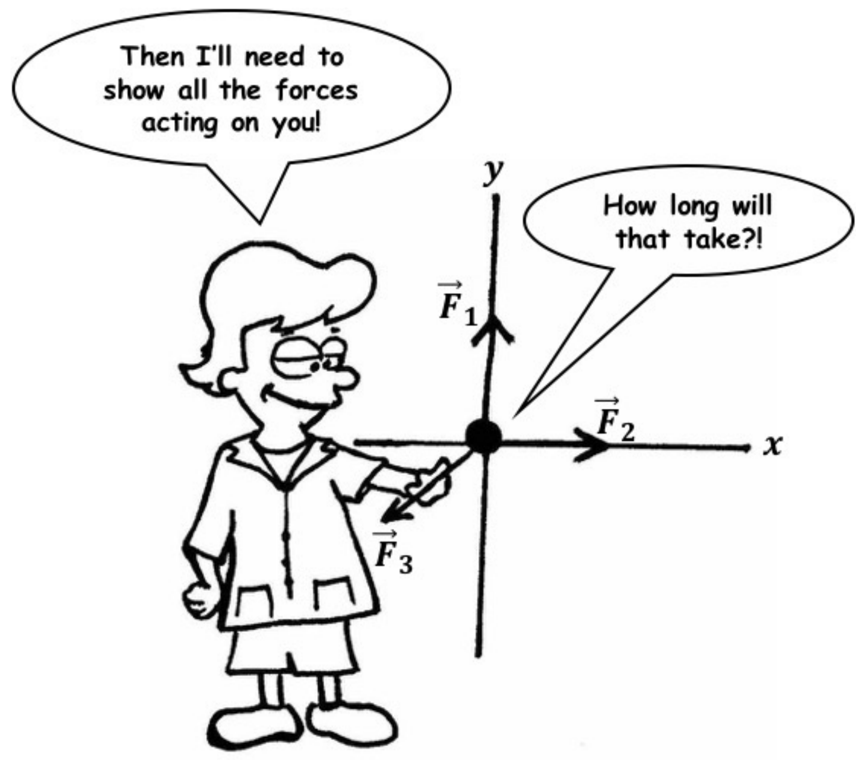
Applying Newton's Second Law:

Consider Sandy who attempts to apply Newton's second law to a block. First the block is considered to be a particle (a dot) and is placed at the origin of a coordinate system. An object can be assumed to behave as a particle if all parts of the object move in exactly the same way.

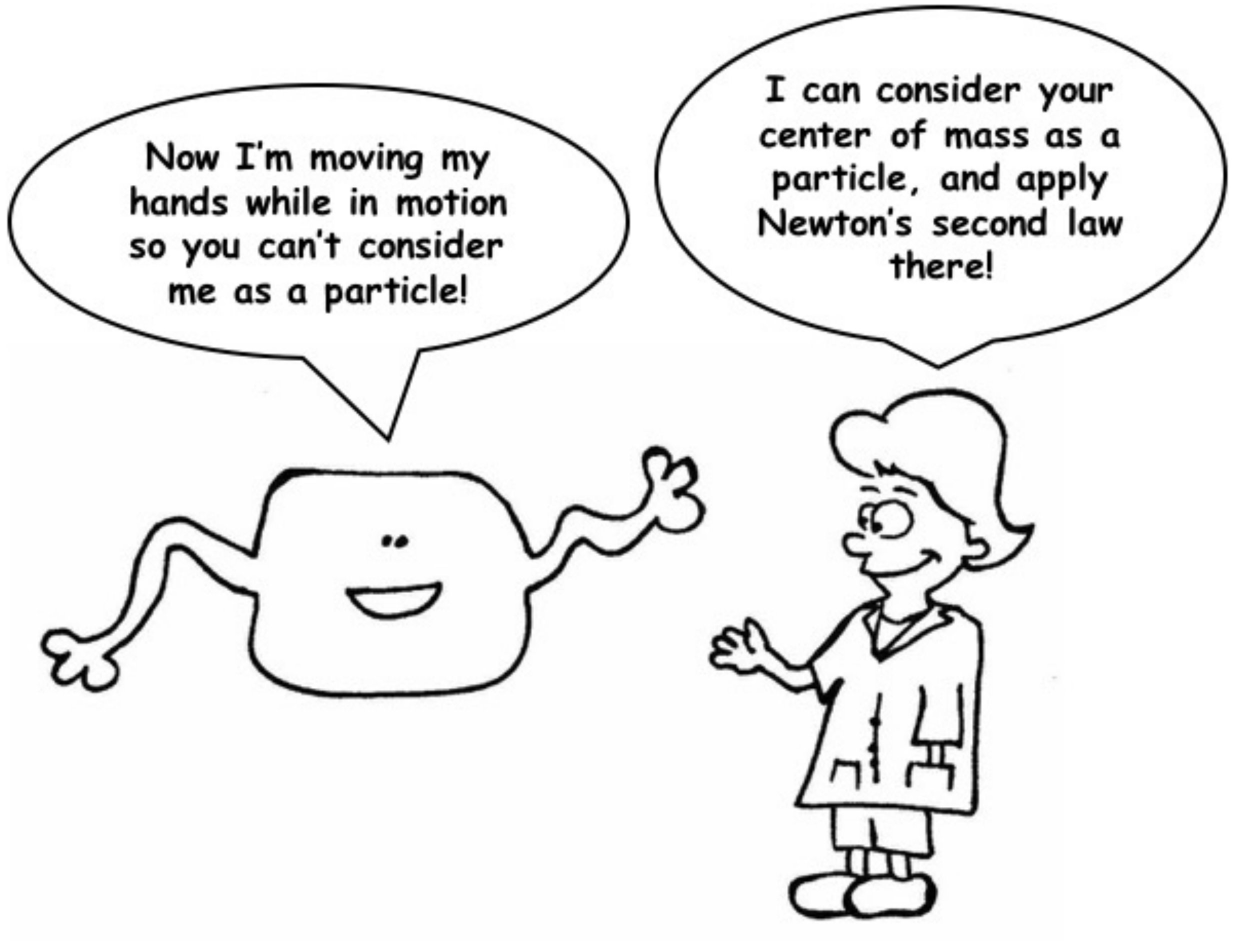




Next all the forces (represented by vectors) acting on the object are shown. This kind of presentation is known as a free-body diagram. The forces in which the object exerts on other objects are not shown in this diagram. Newton's Second law is then applied using the components of the forces along each axis. To simplify the analysis, the coordinate system should be oriented in such a way that some forces are directed along the axes.



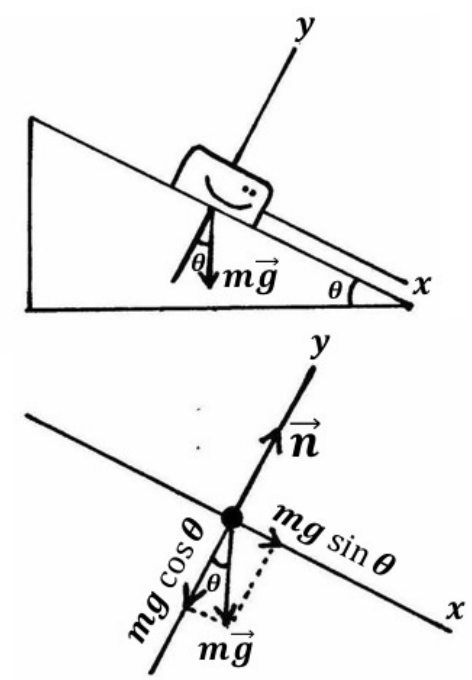
The mass and friction of a pully or a rope are usually neglected. In the case when different parts of the object move in different ways, the object cannot be treated as a particle, instead it must be treated as a system of particles. The motion of the object can then be represented by the motion of its center of mass which behaves as if all of the mass of the object is concentrated there and as if the net external force is applied there (we will discuss more about the center of mass later). Therefore, Newton's second law is applied to the center of mass which will then represent the dot in the free-body diagram.



Consider a block accelerating down a frictionless incline. The free-body diagram of the block is shown. Applying Newton's second law both along the x and y directions gives

$$\sum F_x = mg\sin\theta = ma_x$$

$$\sum F_y = n - mg\cos\theta = 0$$



This gives

$$a_x = g\sin\theta$$

and

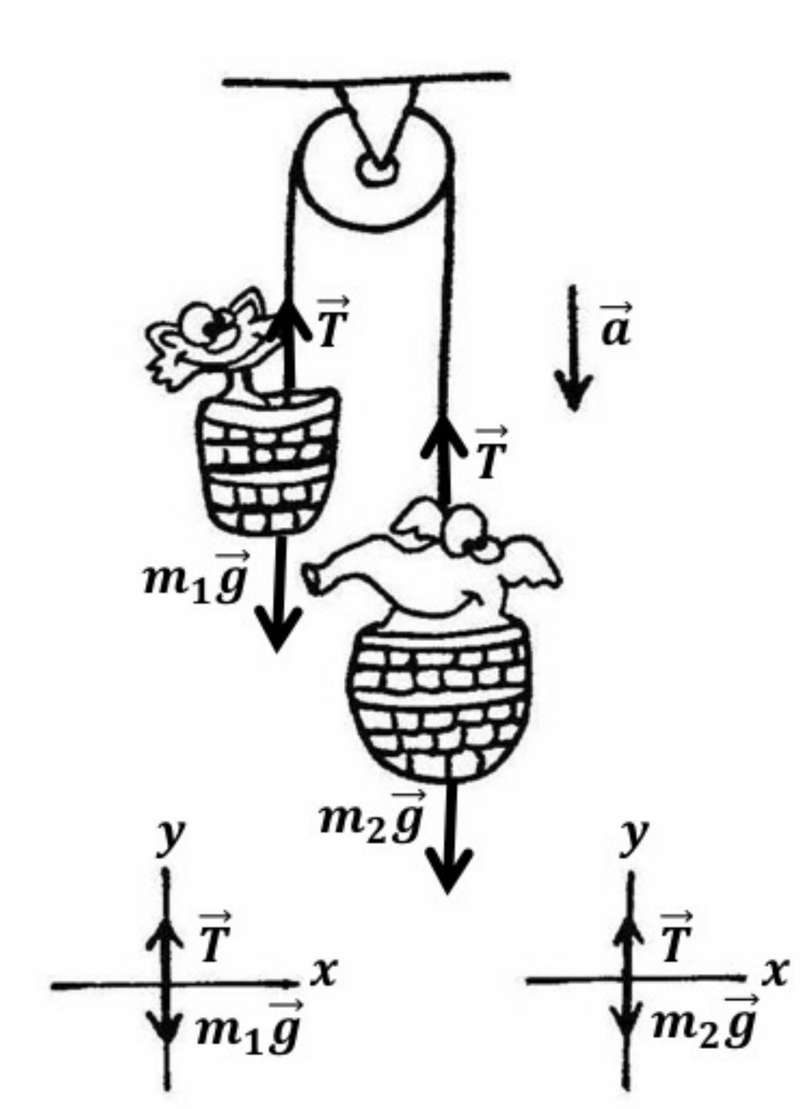
$$n = mg\cos\theta$$

Which shows that the acceleration of any object regardless of its mass on a frictionless surface is $g\sin\theta$. Note that the normal force is less than the weight of the object in this case.

As another example, consider a cat and a baby elephant hung by a light non-stretchable rope over a frictionless pulley of negligible mass. This kind of arrangement is known as an Atwood machine. As expected the baby elephant starts accelerating downwards while the cat accelerates upwards. Because they are connected by a rope, they both have the same acceleration a. The free-body diagrams for each of the cat and the elephant are shown. By Applying Newton's second law in the y-direction for each object, you can find that

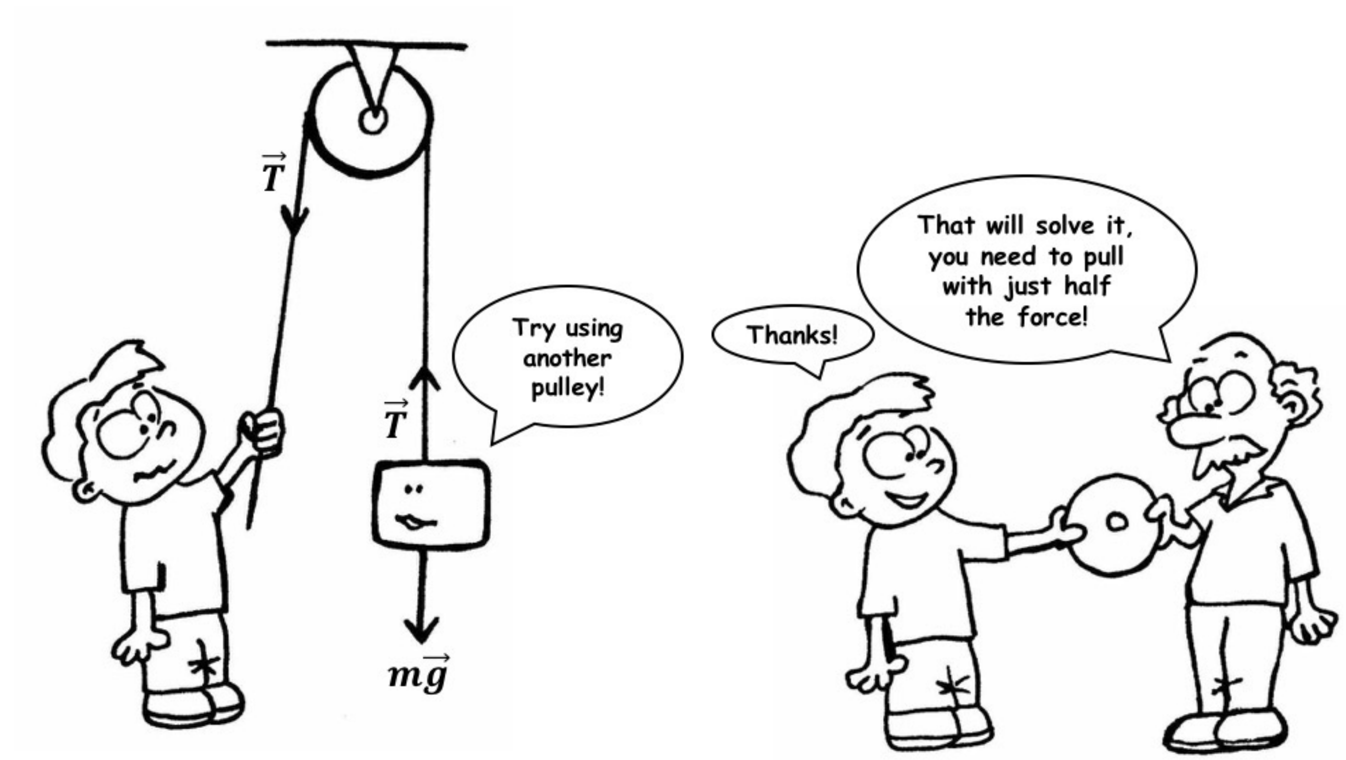
$$a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g$$

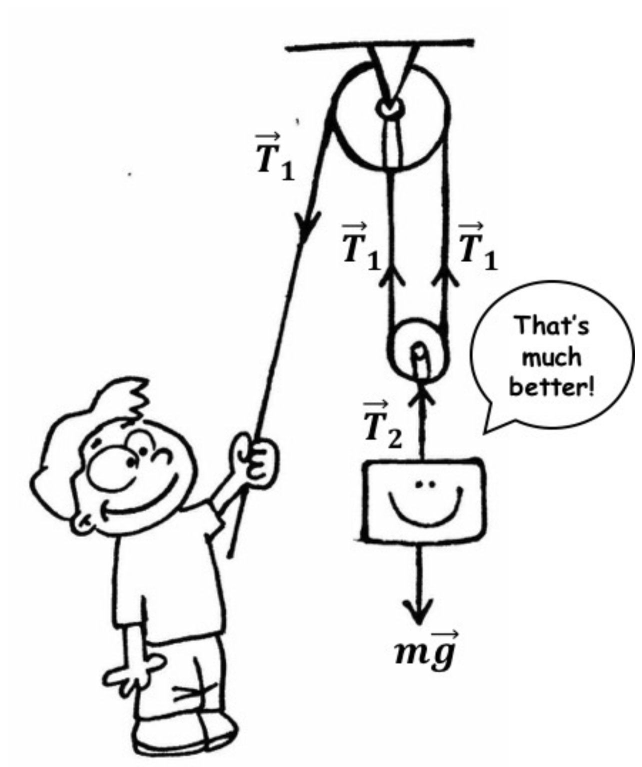
$$T = \left(\frac{2m_1 m_2}{m_2 + m_1}\right) g$$



An important simple machine that can make lifting things up much easier is a pulley. Let's consider this boy who wants to lift a block but is struggling because it's too heavy and his muscles can't supply enough force. The block suggested that he uses an additional pulley that it thinks his grandpa has.

When using one pulley, the minimum force (tension in the rope) needed to lift the block is equal to the block's weight. i.e. $T = w$. When two pulleys are used, $T_2 = w$, and since $2T_1 = T_2$, the boy needs to pull the block with just half the force $T_1 = \frac{w}{2}$.





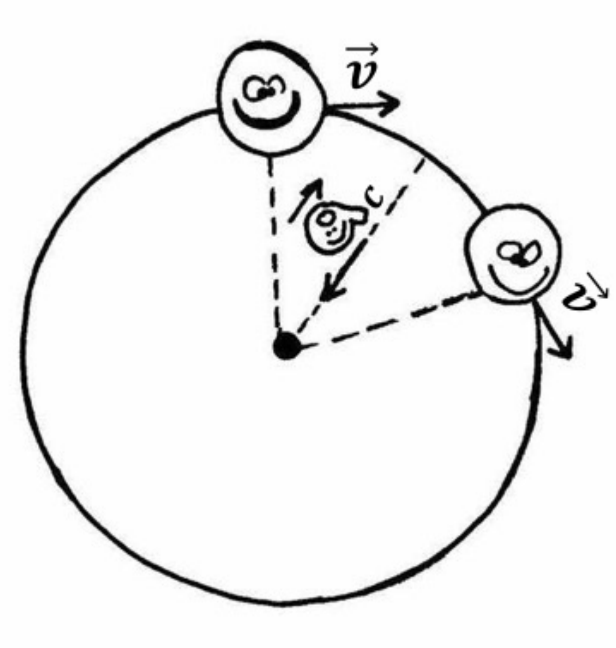
Uniform Circular Motion and the Centripetal Force:

As we have seen earlier, in uniform circular motion, the magnitude of $\vec{v}$ is constant while its direction is changing continuously. The resultant acceleration is known as the centripetal acceleration a_c and is directed towards the center of the circle. From Newton's second law, this centripetal acceleration is caused by the centripetal force F_c

$$F_c = ma_c$$

The centripetal acceleration is given by

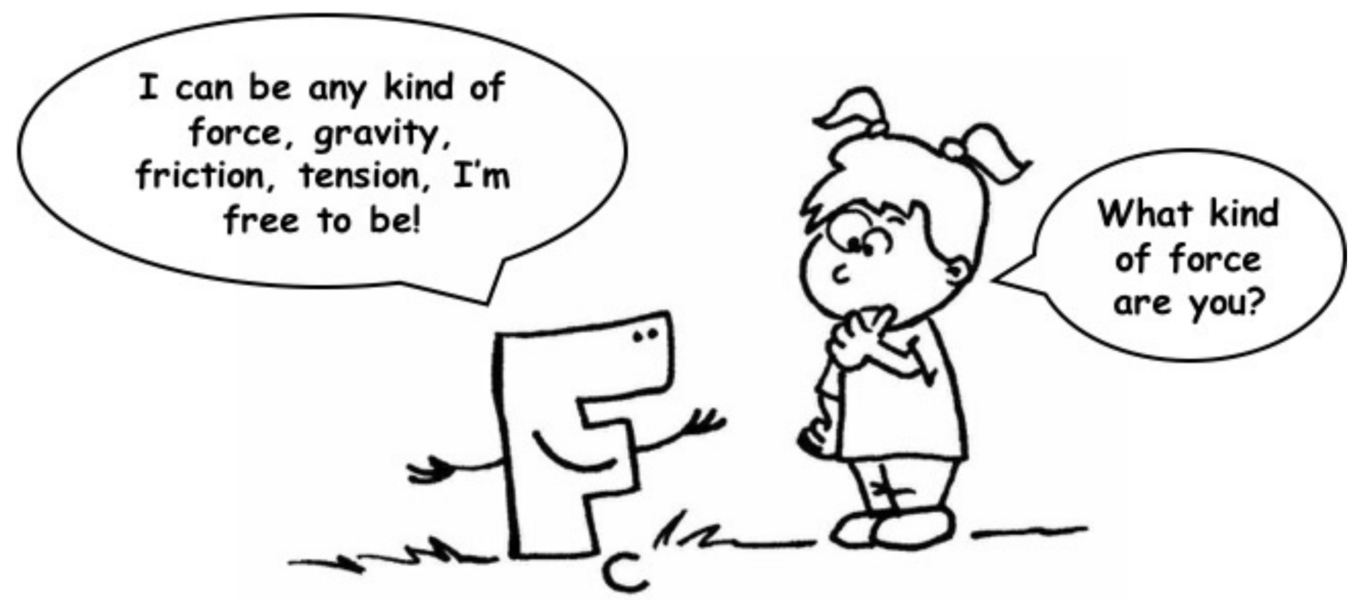
$$a_c = \frac{v^2}{r}$$



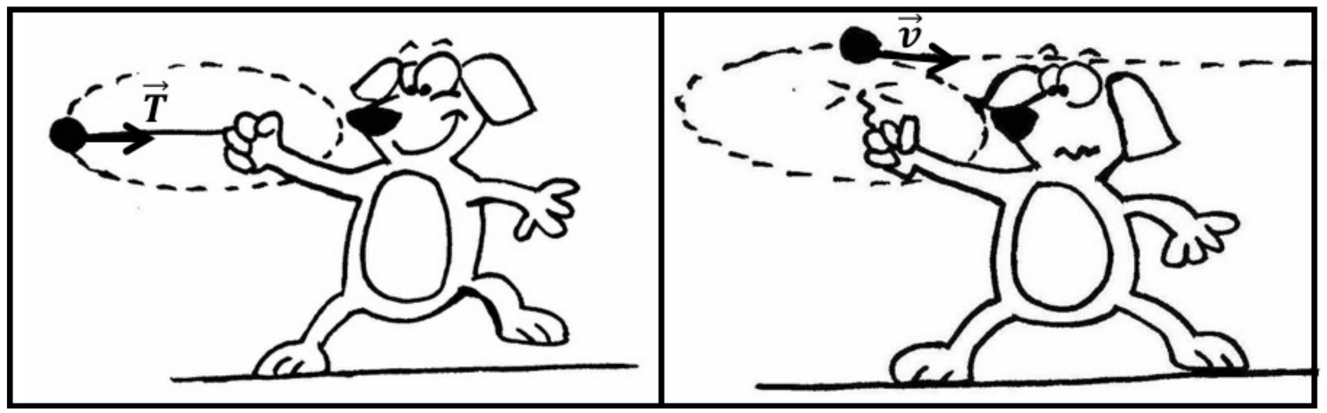
and hence the centripetal force is

$$F_c = \frac{mv^2}{r}$$

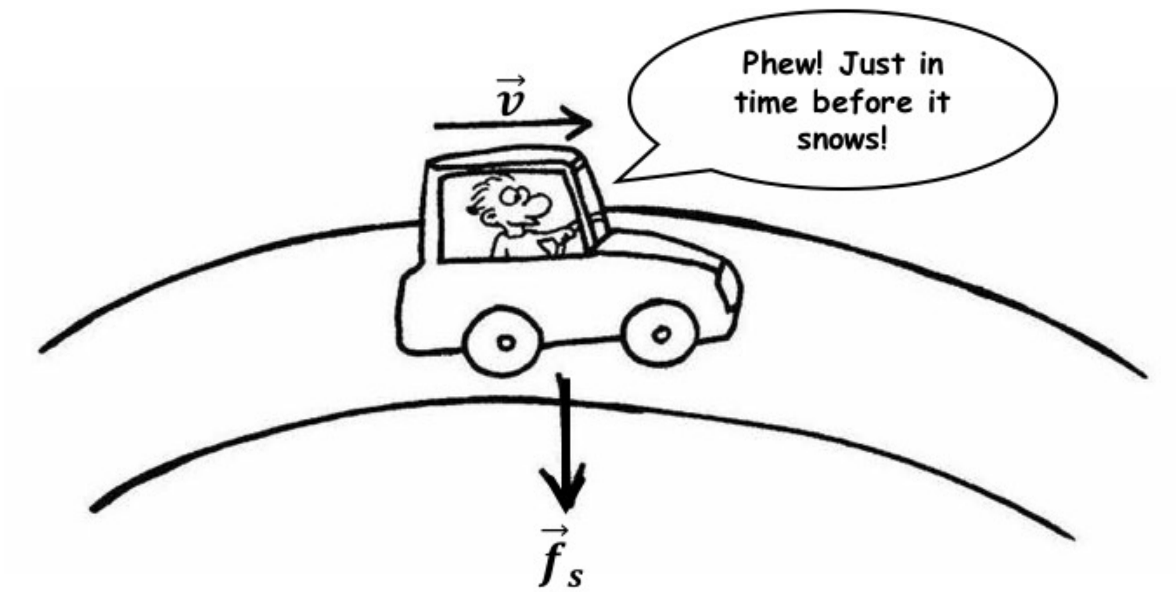
which is the force required to keep an object in its circular path.



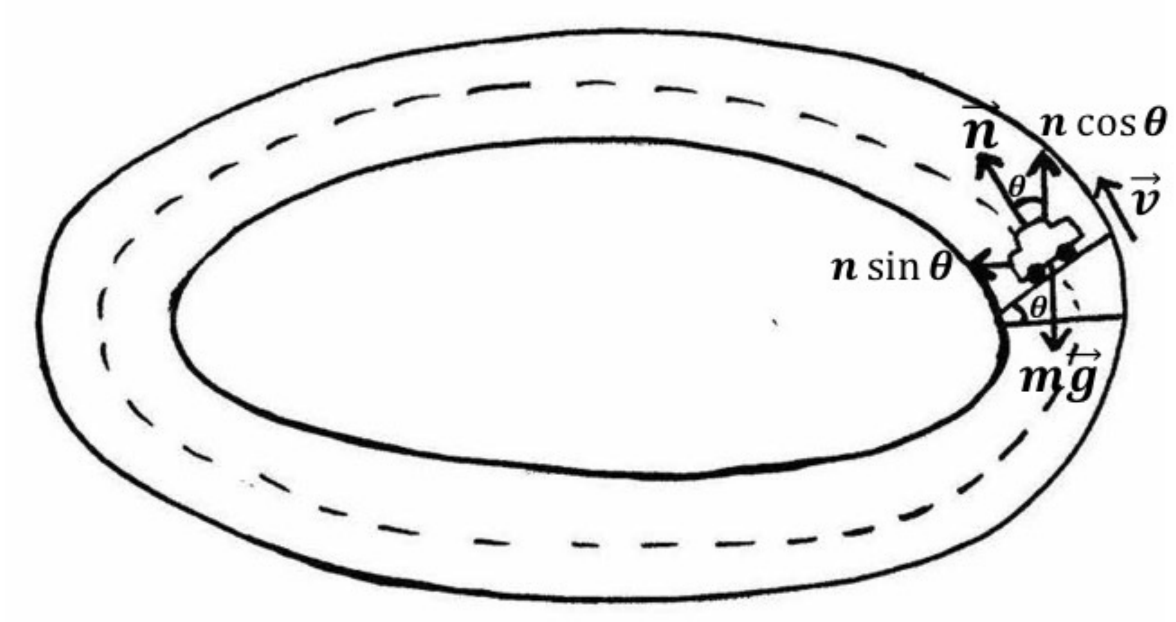
Suppose a ball attached to a string is in uniform circular motion as shown below by our friend Bud. If at any instant $F_c = 0$, i.e. the rope is cut, then the ball will move in a straight line that is tangent to the circle (following Newton's first law) and fall due to gravity.



Let's consider a car taking a turn with a constant speed, the reason it stays in a circular path without skidding is the



statistical frictional force which provides the required centripetal force. If the road is banked at an angle, the horizontal component of the normal force ($n \sin\theta$) as shown below can provide the required centripetal force to keep the car in a circular track without skidding and it is not necessary to depend on friction as in the case of a flat roadway.

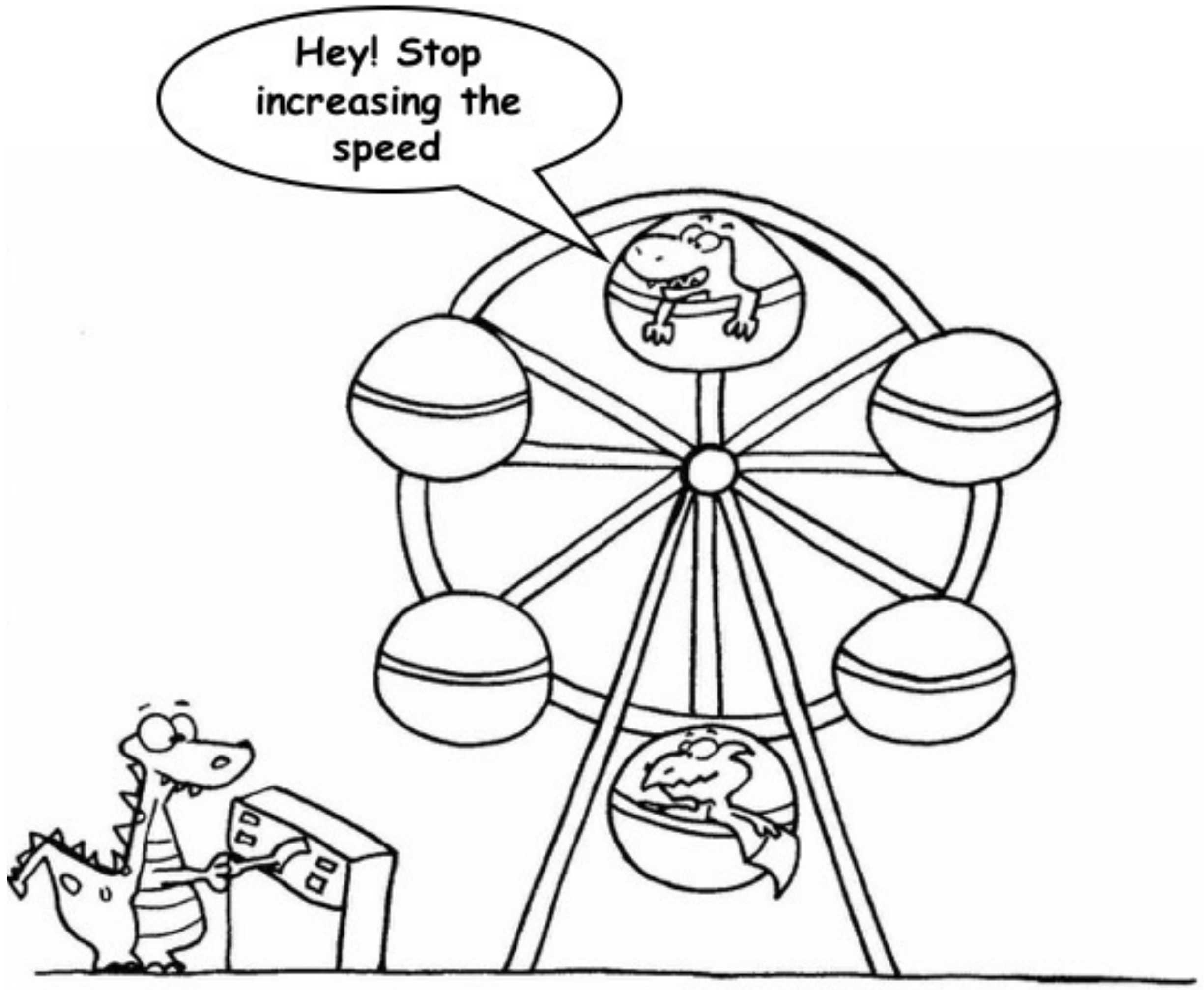


Now let's consider a Ferris wheel where it happened that three dinosaurs escaped from the laboratory at night and went to the amusement park. The dinosaur at the top of the wheel will feel lighter because the normal force exerted on it by the seat there is less than its weight. This can be shown by applying Newton's second law to the top dinosaur as

$$\sum F = n_T - mg = -\frac{mv^2}{r}$$

which gives

$$n_T = m(g - \frac{v^2}{r}) < mg$$



For the dinosaur at the bottom, it will feel heavier since the normal force on it there is larger than its weight as follows

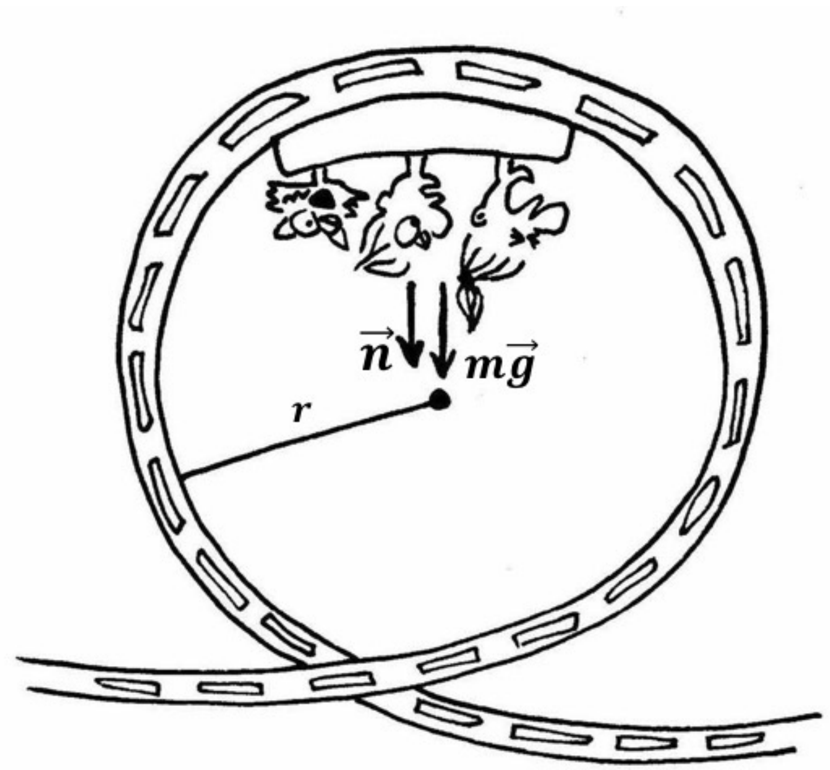
$$\sum F = n_b - mg = \frac{mv^2}{r}$$

which gives

$$n_b = m(g + \frac{v^2}{r}) > mg$$

If the dinosaur in control of the speed (who is clearly having fun with it despite the pleadings from the top dinosaur) increases it such that mv^2/r becomes larger than mg, then n_T becomes negative which is meaningless. In this case the top dinosaur will lose contact with the seat unless it has its seatbelt on.

As another example, consider the roller coaster where we want to calculate the minimum speed for the car to take the loop safely without falling off the track at the top.



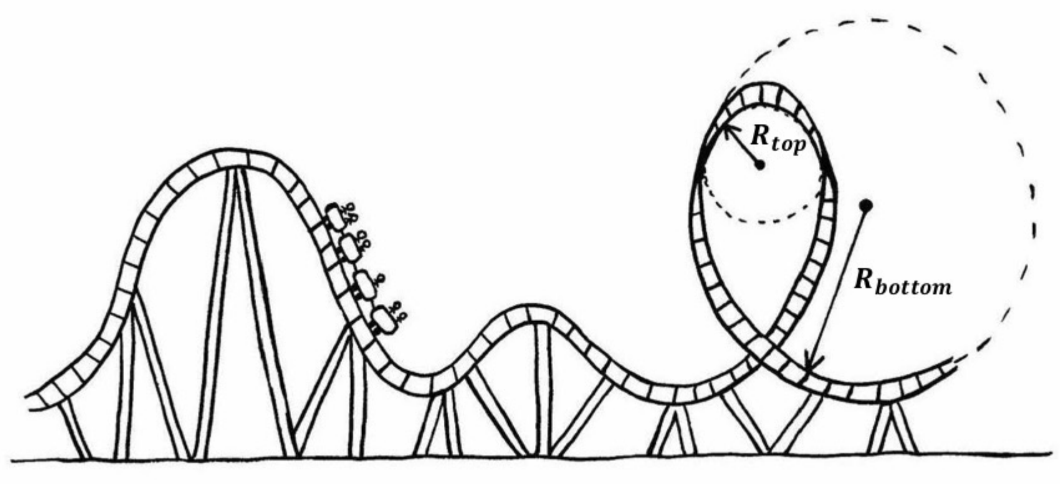
By applying Newton's second law on any rider when the car is at the top we have

$$n + mg = \frac{mv^2}{r}$$

When the rider is barely in contact, then $n = 0$ and therefore the minimum speed to loop the loop is

$$v = \sqrt{rg}$$

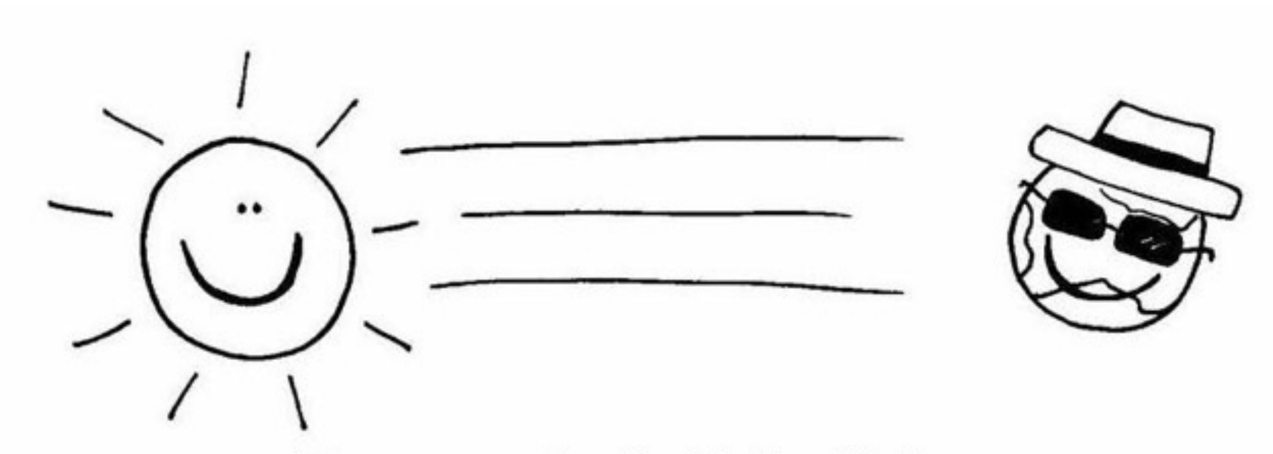
Note that in real life, roller coaster loops are never circular but are shaped like a teardrop. This is because in a circular loop, the force exerted on the rider at the bottom is large when the car is fastest, due to the tight radius of the loop. However, for a teardrop shaped loop, the force at the bottom, when the car is fastest, is reduced due to the large radius of curvature of the track there. While at the top, the radius of curvature is made smaller so that the riders would feel the same amount of force throughout the loop. This allows the ride to be smoother and safer for the passengers.



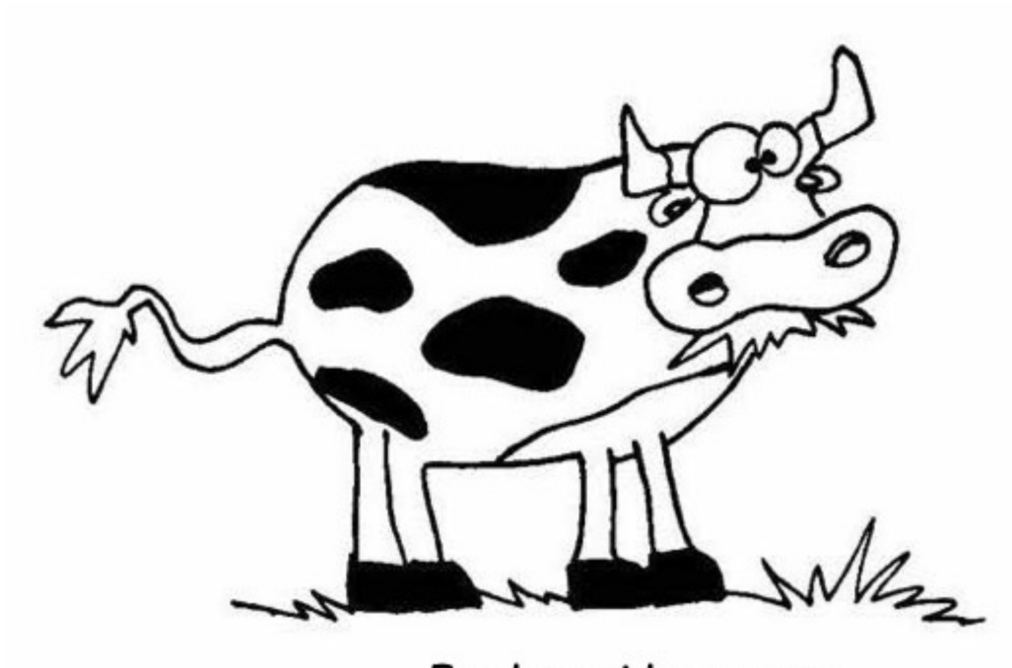
Work and Energy

Work and Energy:

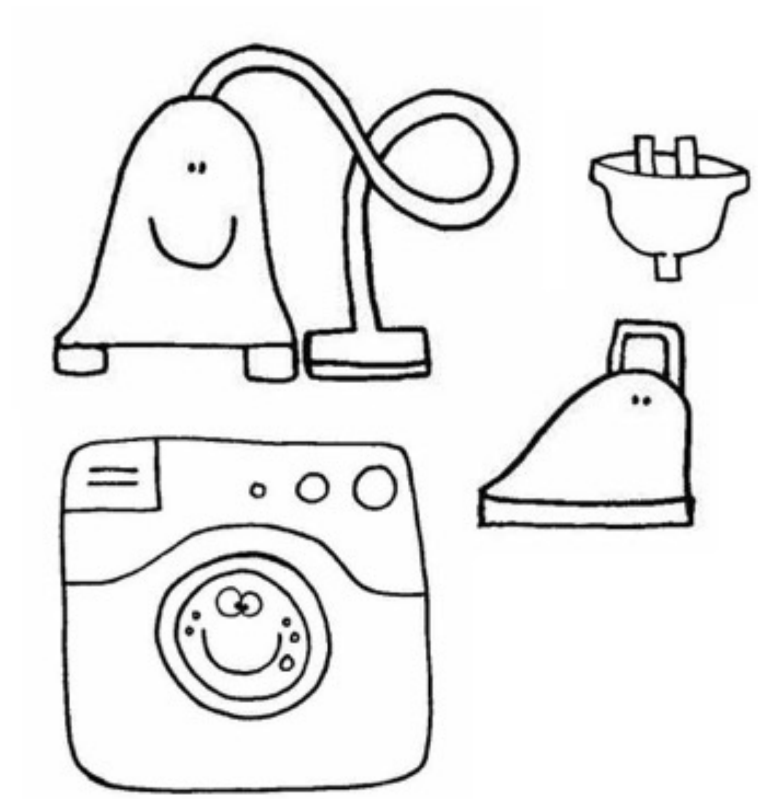
Energy is one of the most important concepts in science. Everything needs energy to function and according to Einstein's famous equation, $E = mc^2$, even mass is just a manifestation of energy.



The sun provides the Earth with the energy needed for the existence of life!



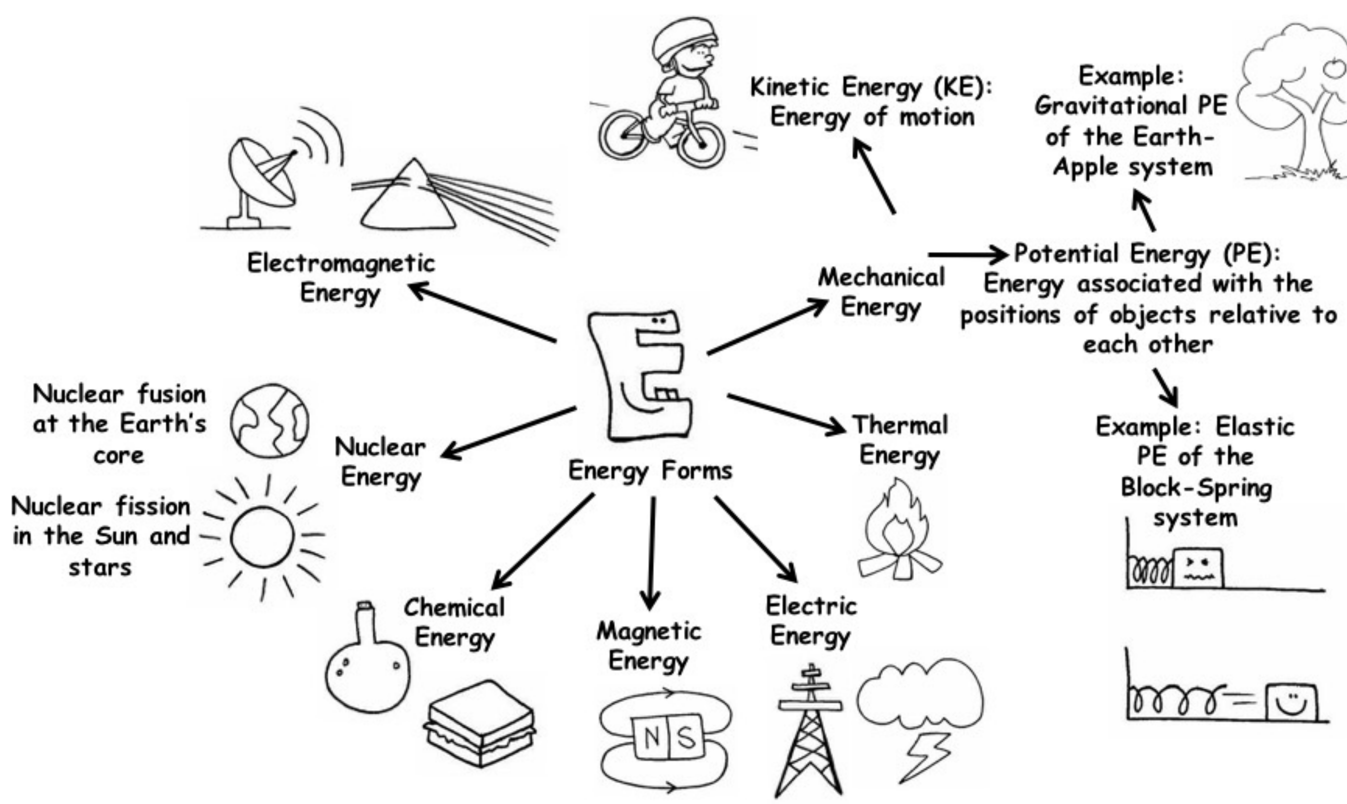
Food provides energy



Electricity provides home appliances with the energy needed to function

Fuel provides a car with the energy needed to move

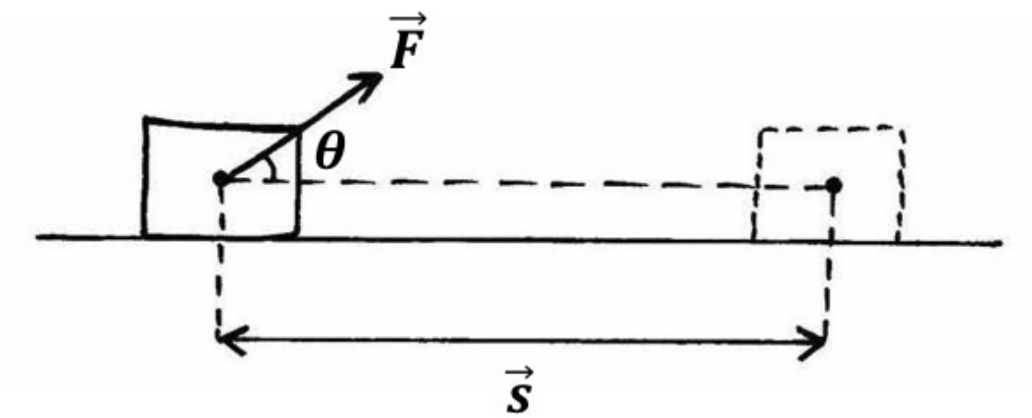
Through experiments, energy was found to be a scalar quantity that can exist in various forms as shown below. In addition, energy cannot be created or destroyed but can only be transformed from one form to another. In other words, the law of conservation of energy states that in an isolated system, the total amount of energy remains constant (conserved) at all times. There are several mechanisms to transform energy, or transfer it in or out of a system, such as through work, heat and matter transfer.



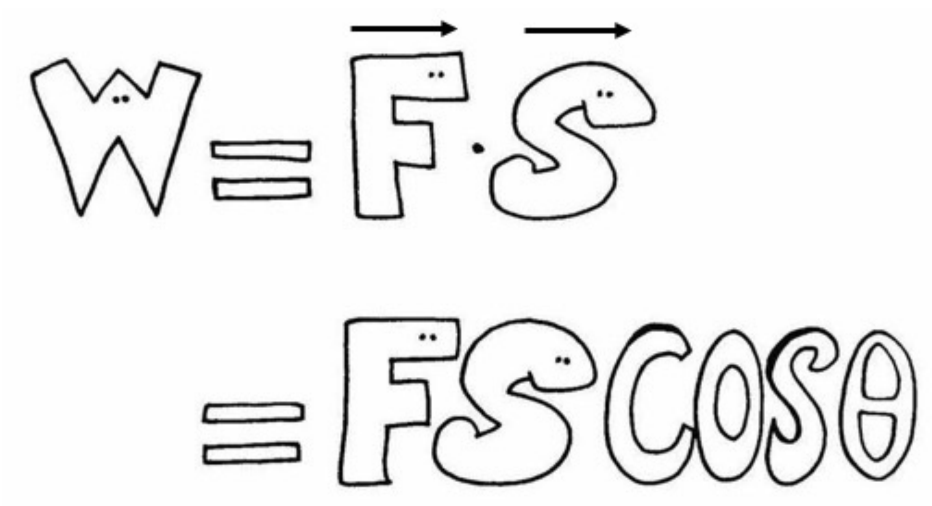
Work:

In everyday life, the term work can be used in many different contexts, but it can generally mean that energy was spent somewhere in doing something, like in carrying a box (the chemical energy in the body is converted into microscopic muscular motions), mowing the lawn or even psychological or mental work. In physics, however, work has a definite meaning.

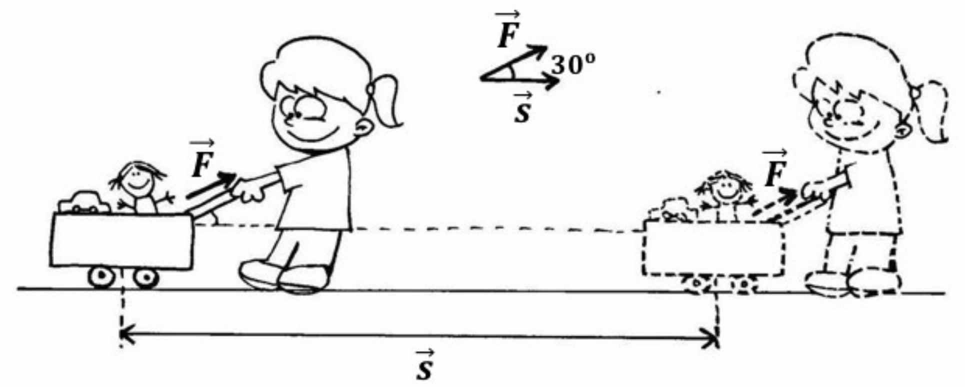
To understand it, consider a block that undergoes a displacement $\vec{s}$ along a straight line while being acted upon by an external constant force $\vec{F}$ which makes an angle θ to $\vec{s}$.

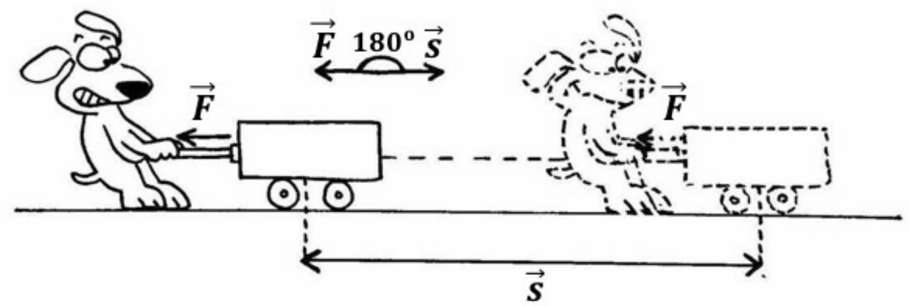


The work W done on the object is then defined as the scalar (dot) product of $\vec{F}$ and $\vec{s}$ as follows



This work represents a transfer of energy to or from the object via that force. So, if $W > 0$ (such as in the first case below when $\theta = 30^{\circ}$) energy is transferred to the object and if $W < 0$ (such as if $\theta = 180^{\circ}$ when Bud was trying to stop the wagon) energy is transferred from the object.





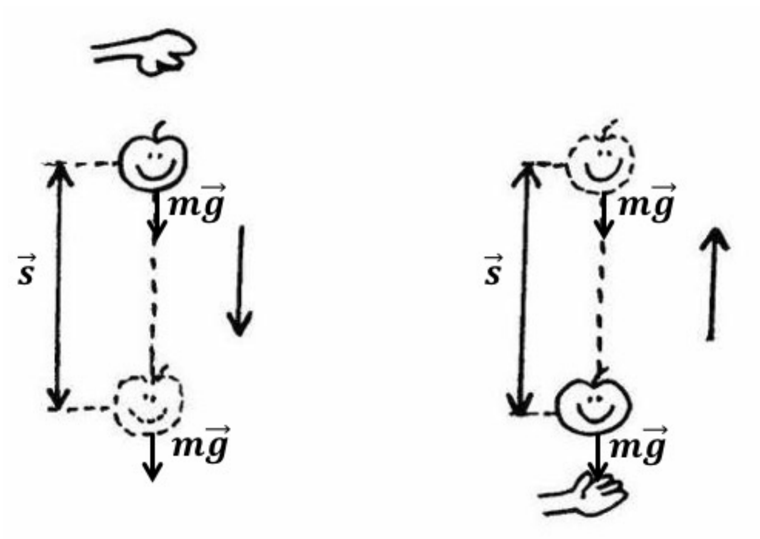
This also shows that if there is no displacement, no work is done. Therefore, if you apply a force on an object (such as pushing a wall) but it does not move, then the work done by that force on the object is zero.

The SI unit of work is the Joule (J). One Joule is the amount of energy that a force of one Newton does over a displacement of one meter. Note that the Joule is also the unit of energy.

$$1\,\mathrm{J} = 1\,\mathrm{kg.\,m^2/s^2}$$

Now let's consider the work done by gravity as an apple falls. Since the force of gravity is in the same direction as the displacement, the work done is positive ($W =$

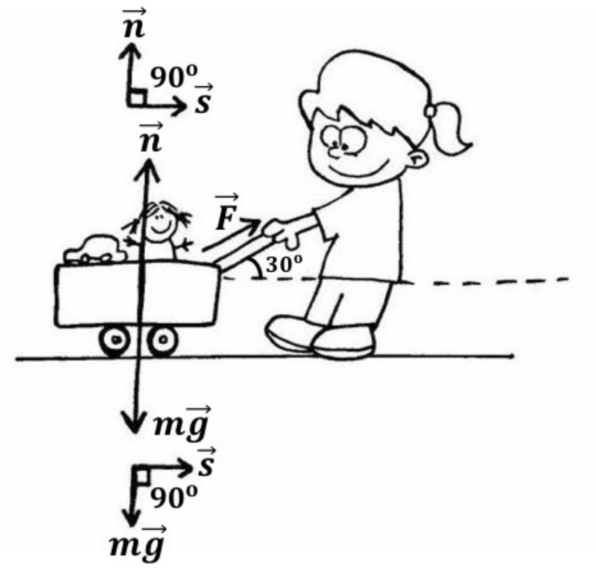
$mgs\cos 0^{\circ} = mgs > 0$) and the apple accelerates, while if it moves upwards, the work done by gravity is negative ($W = mgs\cos 180^{\circ} = -mgs < 0$) and it decelerates.



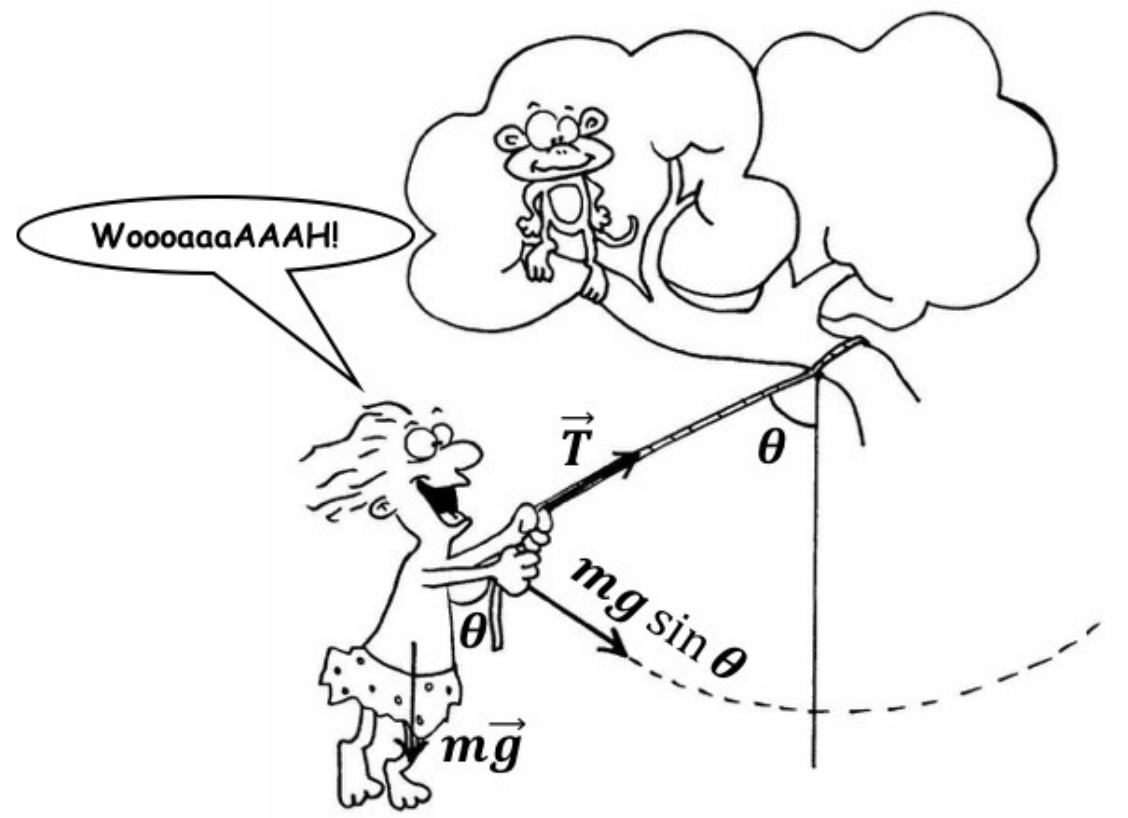
Note that it follows from the definition of work that the work done by a force on an object is zero if it is perpendicular to the displacement. For example, the work done by the normal force $\vec{n}$ or the force of gravity $m\vec{g}$ as the wagon is displaced is zero since both are perpendicular to the displacement at all times

$$W_n = ns\cos 90^{\circ} = 0$$

$$W_g = mgs\cos 90^{\circ} = 0$$



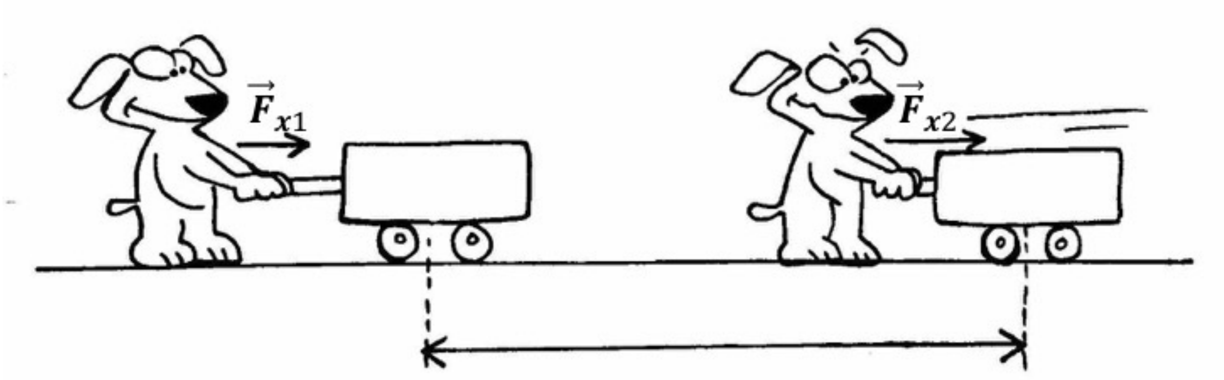
Another example is the work done by the tension force $\vec{T}$ of the rope used by Tarzan, which is zero at all times since it is perpendicular to the displacement at every point. It is



the horizontal component of gravity ($mg\sin\theta$) that is doing the work, while $\vec{T}$ is providing the centripetal force needed to keep Tarzan in his circular path.

Work Done by a Varying Force:

Suppose that Bud decides to change the magnitude of the force he applies to the wagon as it is displaced while keeping the direction of that force constant.

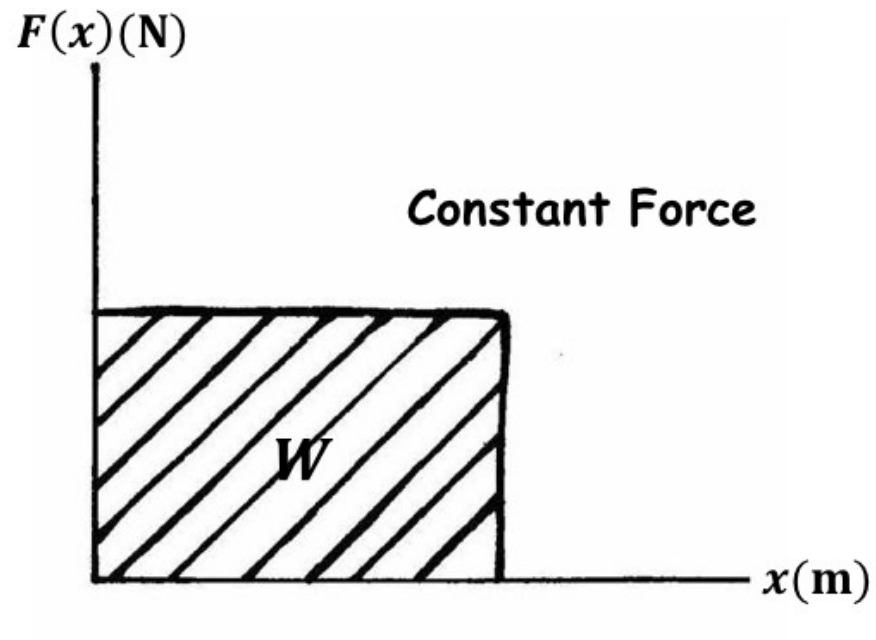


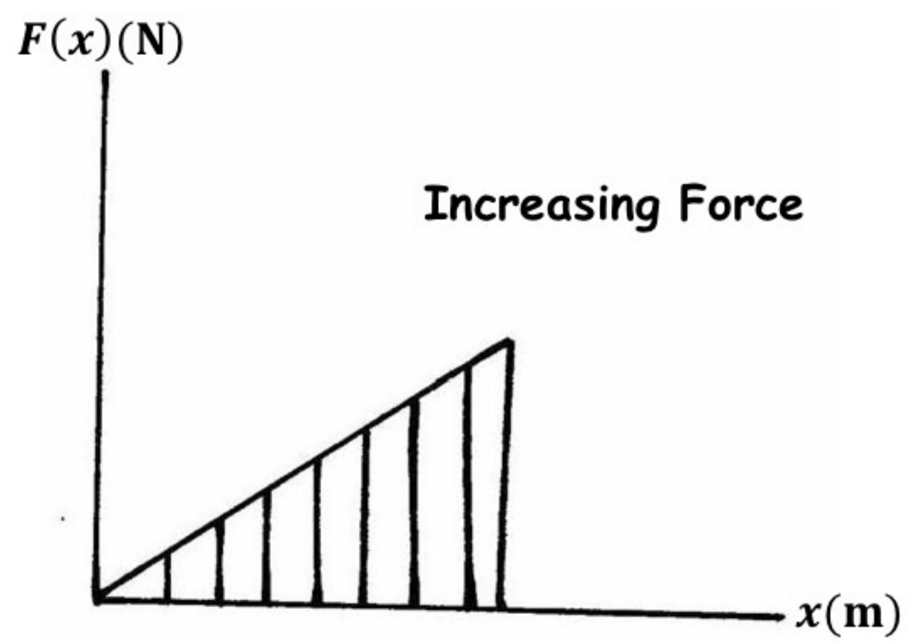
The work is then given by

$$W = \int_{x_i}^{x_f} F(x)dx$$

$$= \text{The area under the curve of the F(x) vs x graph}$$

The graphs below show how the force changes with displacement in the case of a constant force and for a force that varies linearly with displacement. The total area under the curve for a certain displacement represents the work done.





Kinetic Energy (KE):

Kinetic energy is the energy associated with the motion of an object. Its unit is Joul and it is defined as

$$K = \frac{1}{2} m v^2$$

where m is the mass of the object and v is its speed. As an example, if the mass of the rabbit is the same as that of the turtle, then the rabbit has a higher kinetic energy.

The Work-Energy Theorem:

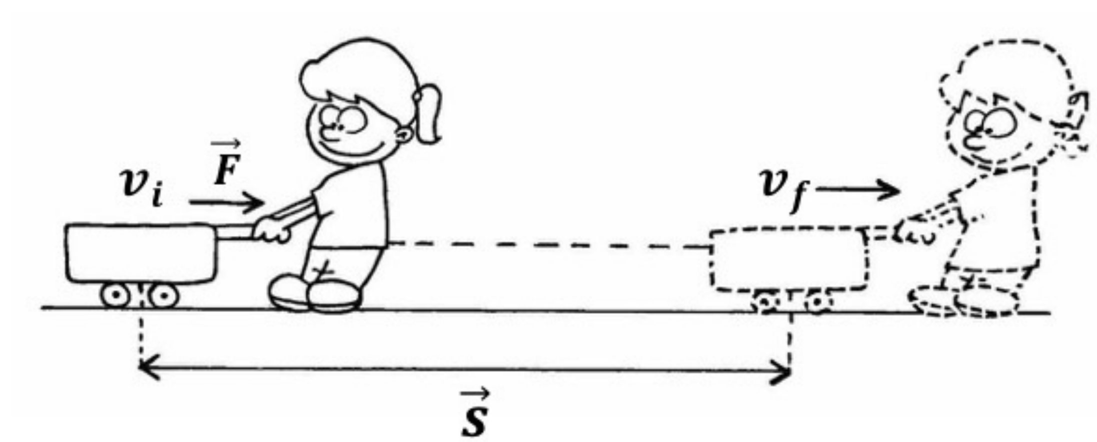
The work-energy theorem states that the work done by the net force in displacing a particle is equal to the change in the KE of the particle. Note that by particle we mean a particle-like object or its center of mass if it cannot be treated as a particle. This can be shown through an easy derivation as follows

$$W = \int_{x_i}^{x_f} F(x)dx = \int_{x_i}^{x_f} m\frac{dv}{dt}dx = \int_{x_i}^{x_f} m\frac{dv}{dx}\frac{dx}{dt}dx$$

$$W = \int_{v_i}^{v_f} mvdv$$

which gives

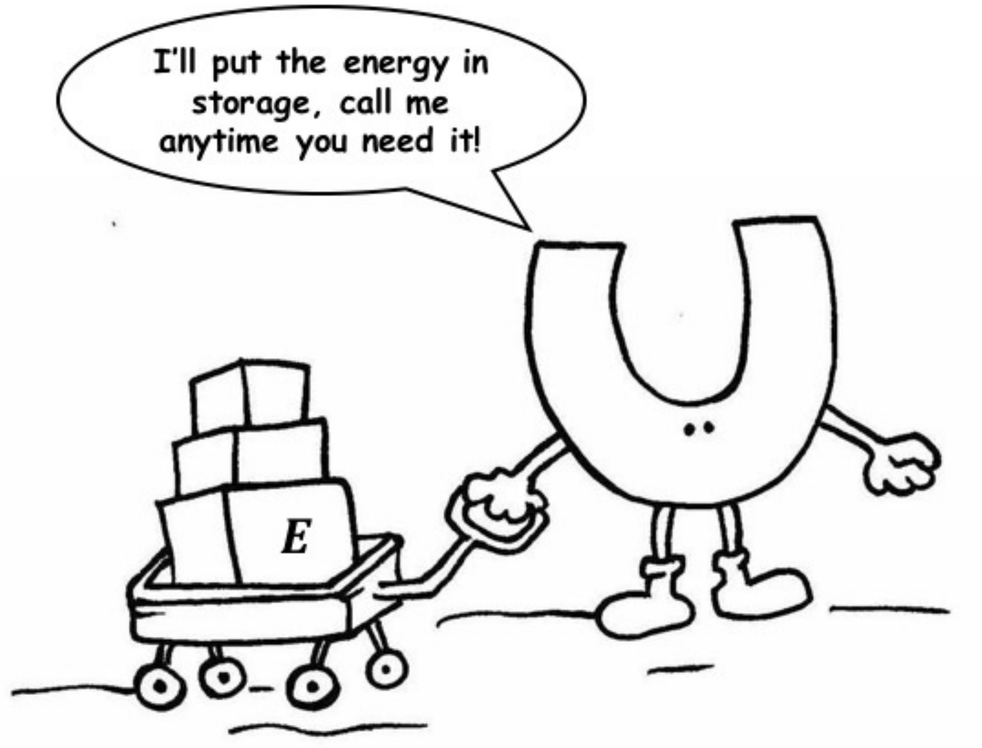
$$W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$



Depending on whether the work done is positive or negative, the kinetic energy of the object increases or decreases respectively.

Potential Energy (PE):

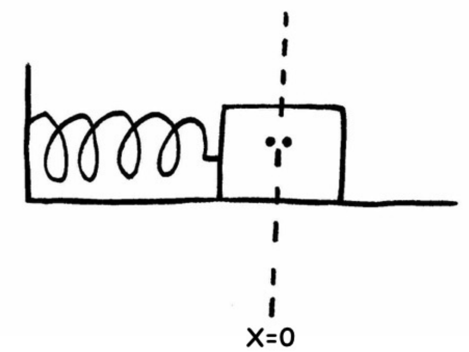
For a system of two or more objects, the PE (U in unit Joul) of the system is the energy associated with the positions of the objects relative to each other. As we shall see later, this energy is defined only in terms of a conservative force. It is called potential energy because it is an energy that is stored in the system and can be convert into KE and back into PE without the total energy of the system being lost, i.e. the total mechanical energy of the system is conserved (as we shall see later).



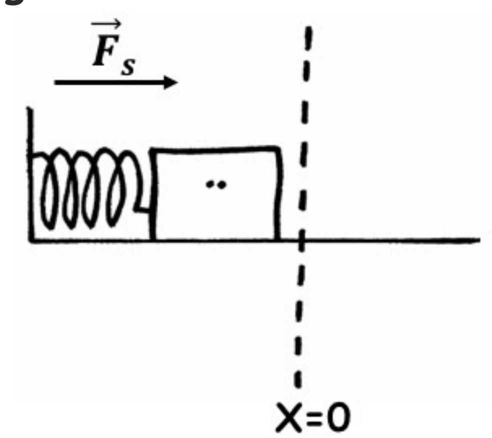
The Elastic Potential Energy:

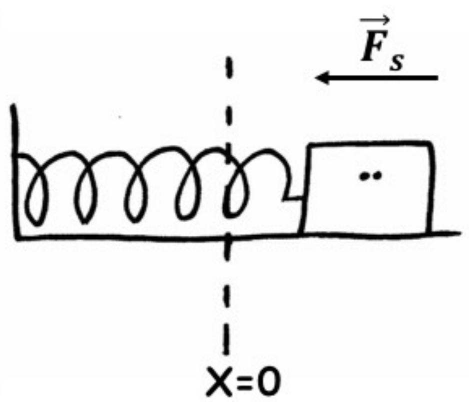
Consider a block attached to a light spring. If the block is displaced a small distance to the right or left of the equilibrium position $x = 0$, then according to Hook's law the spring will exert a restoring force on the block that is opposite to its displacement and is given by

$$F_s = -kx$$



where k is the spring constant in N/m. If the block is displaced to the right, the spring will exert a force on it to the left towards the equilibrium position $x = 0$ and vice versa. This force increases with the displacement so it's a varying force.



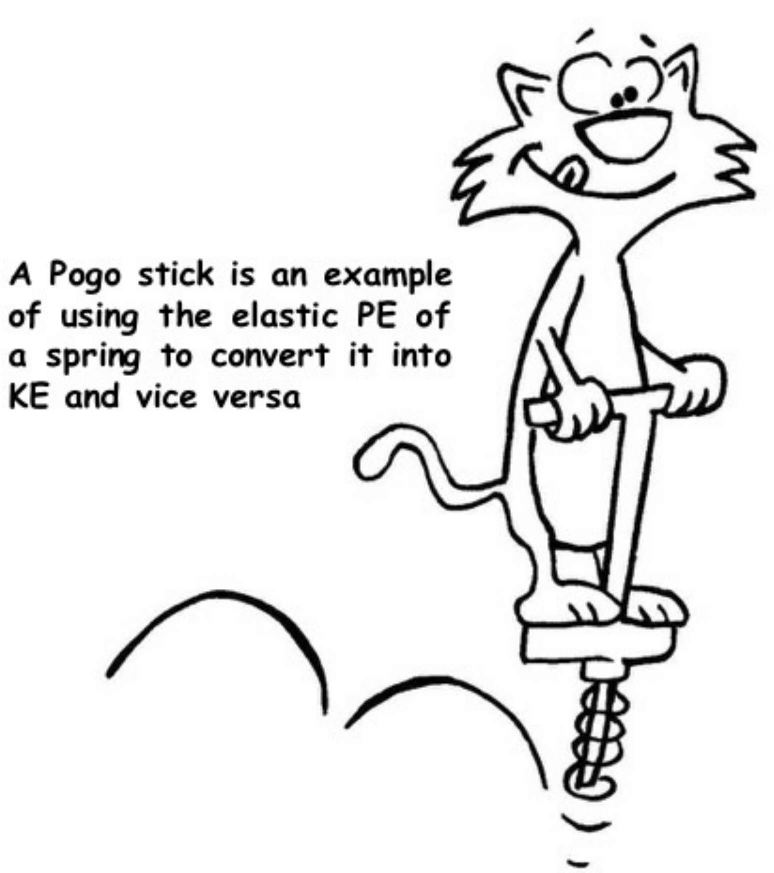


The work done by this force as the block moves from x_i to x_f is

$$W = \int_{x_i}^{x_f} F_x\, dx = \int_{x_i}^{x_f} (-kx)\, dx$$
$$\Rightarrow W = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

The PE of the block-spring system is defined as

$$U_s = \frac{1}{2}kx^2$$



Therefore, the work done on the block by the spring is related to the change in the PE of the mass-spring system through

$$W = U_{si} - U_{sf} = -\Delta U_s$$

Since we also know that $W = \Delta K$, then it follows that this change in PE is accompanied by a change in the KE of the system, and so as one form of energy increases, the other one decreases and vice versa. Therefore, PE is the energy stored in the system and the total energy of the system is not dissipated.

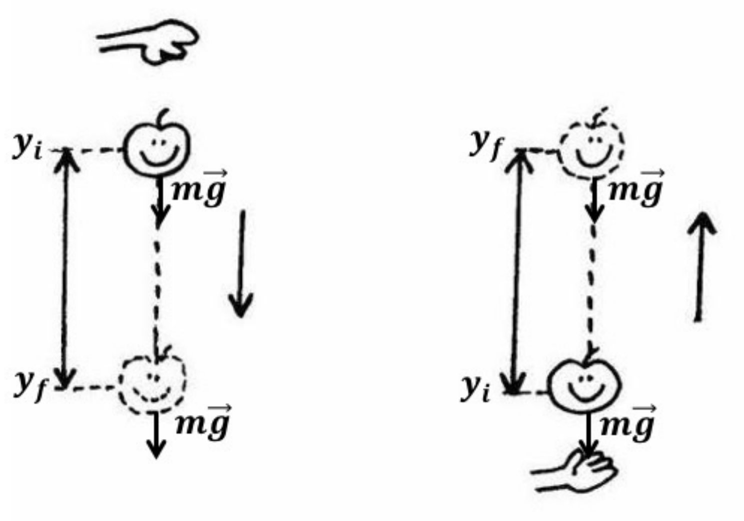
For example, when the boy opened this unpleasant surprise, the PE stored in the spring-clown system is converted into KE and the clown moves upwards.

The Gravitational Potential Energy:

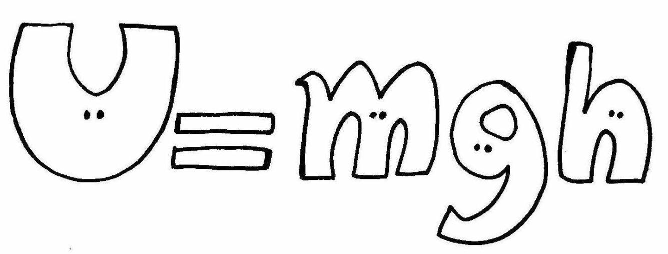
Going back to our happy apple moving under gravity. The work done by the force of gravity $m\vec{g}$ as the apple moves from y_i to y_f, whether falling or moving upwards, is

$$W = \int_{y_i}^{y_f} F_y \, dy = \int_{y_i}^{y_f} (-mg) \, dy$$

$$\Rightarrow W = mgy_i - mgy_f$$



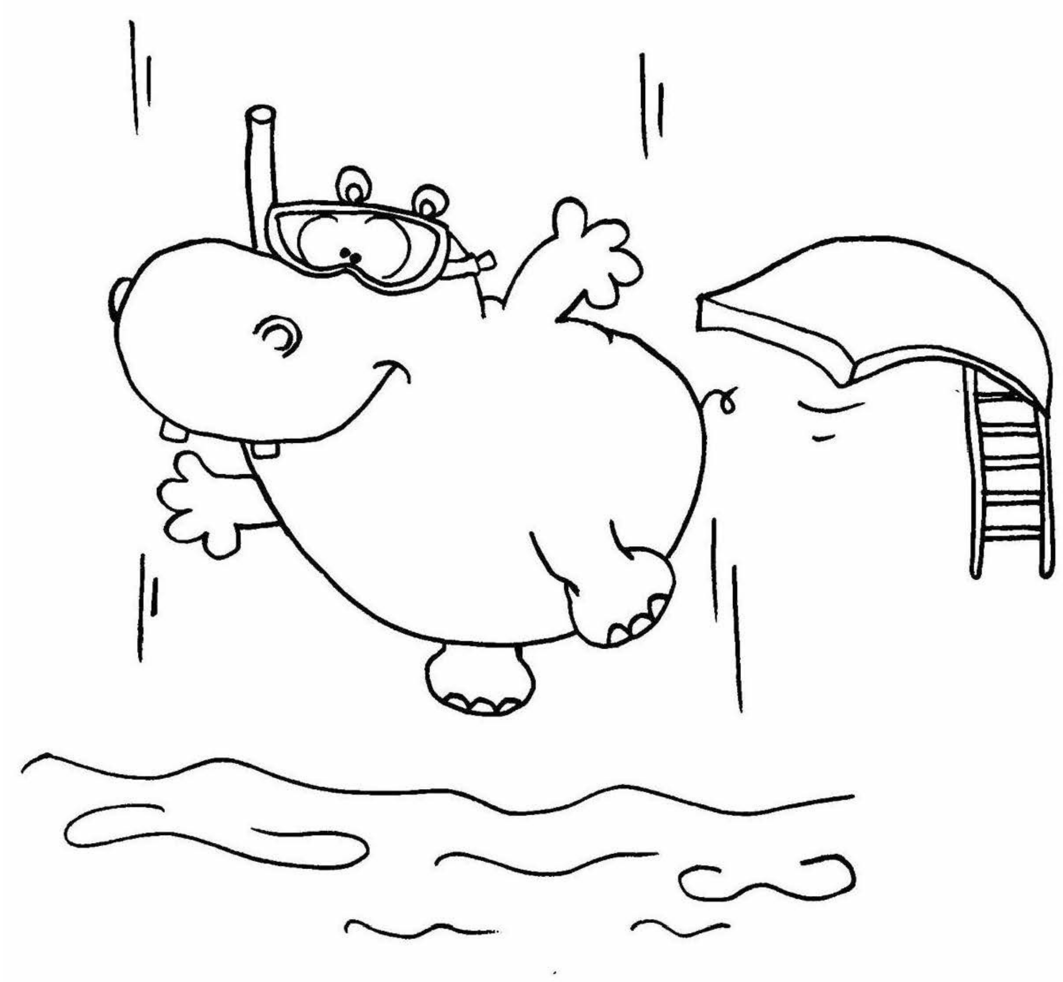
The gravitational potential energy is defined as



And so

$$W = U_{gi} - U_{gf} = -\Delta U_g$$

When the apple falls, the KE of the apple increases while its PE decreases, and if it is thrown upwards, its KE decreases continuously converting into PE. The surface of the Earth is usually taken as the reference point $y = 0$, but it does not really matter where it is chosen since ΔU_gremains the same.

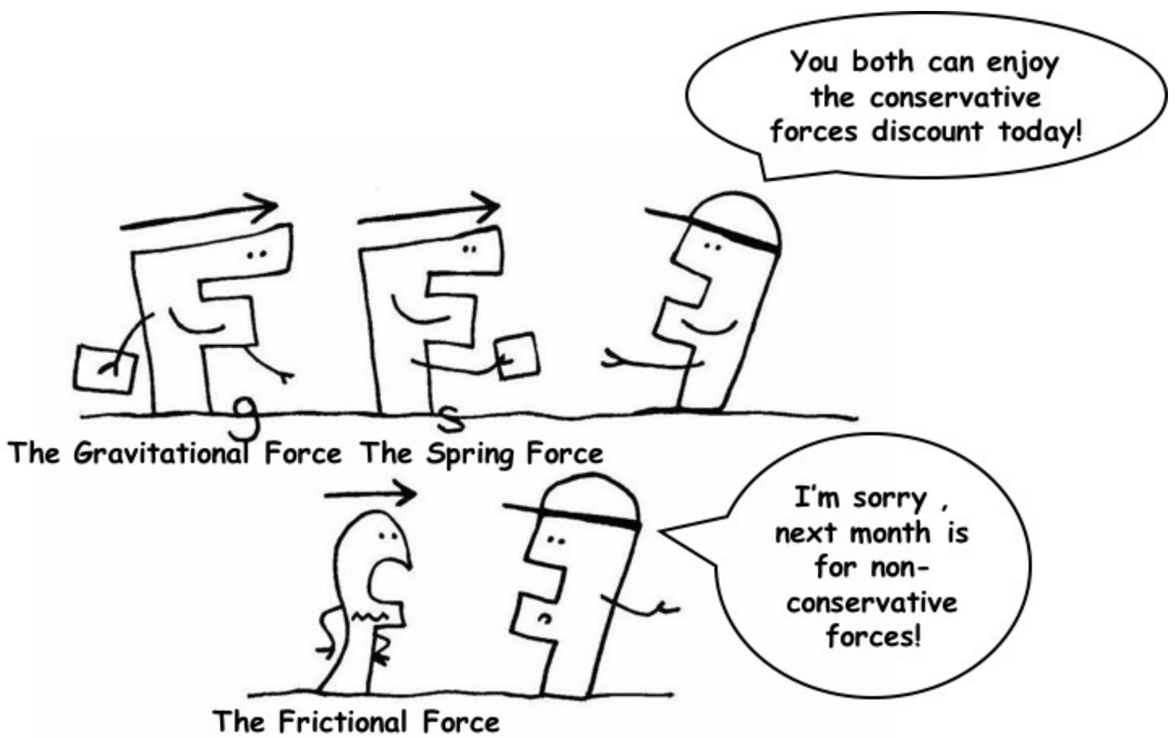


Both the spring force and the force of gravity are conservative forces, since if they do work on an object and change its KE, this change is stored in the (block-spring) or (object-Earth) systems in the form of the PE that is associated with that force. Therefore, conservative forces do not dissipate the energy of the system, instead they conserve it.

The properties of a conservative force are:

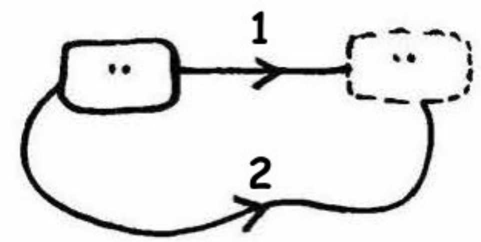
1-The work done by a conservative force is independent of the path taken by the object, instead it depends only on the initial and final positions.

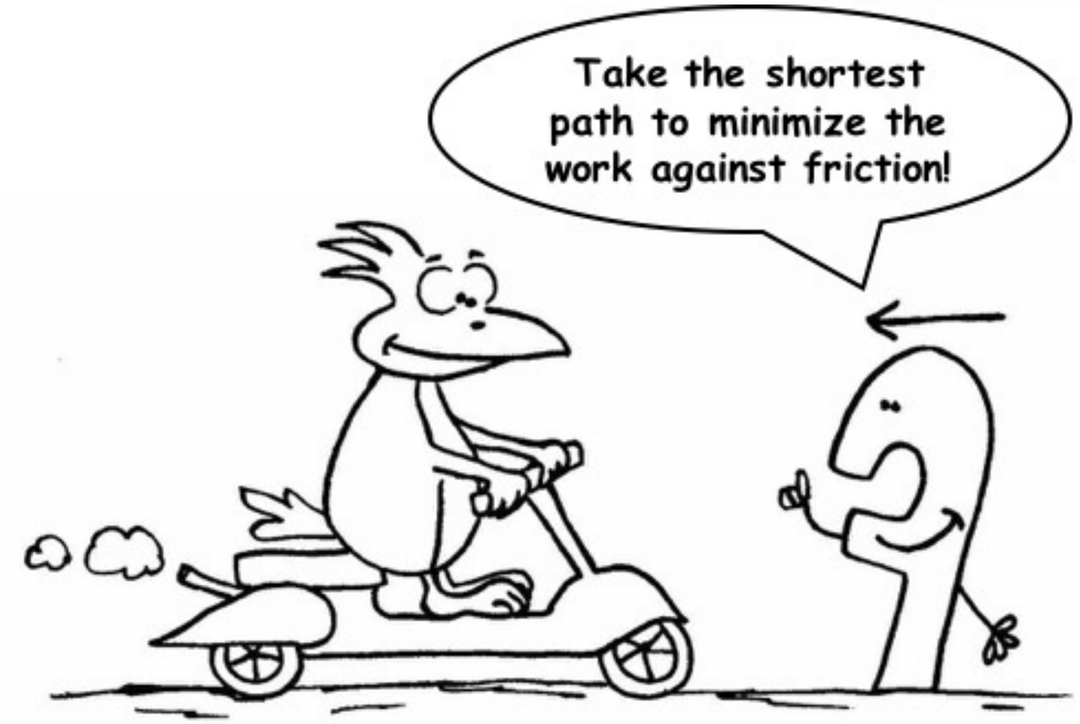
2-The total work done around a closed path is zero.



Using the equations $W = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$ and $W = mgy_i - mgy_f$, it is clear that both the spring force and the force of gravity qualify as conservative forces.

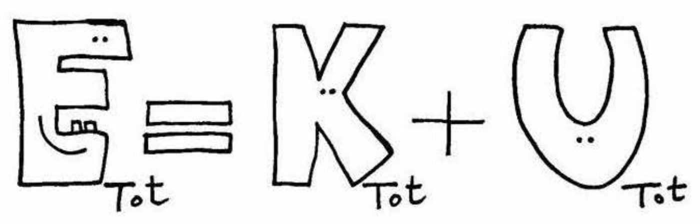
Friction is an example of a non-conservative force. This is because the work done against the force of kinetic friction is more when the block moves along path 2 than along 1, and therefore the work is not independent of the path. The KE of the block in this case is not transformed into PE of the block-surface system, instead it is transformed into internal energy of both the block and the surface. This energy cannot be retrieved back to KE and therefore the total mechanical energy of the system is not conserved.

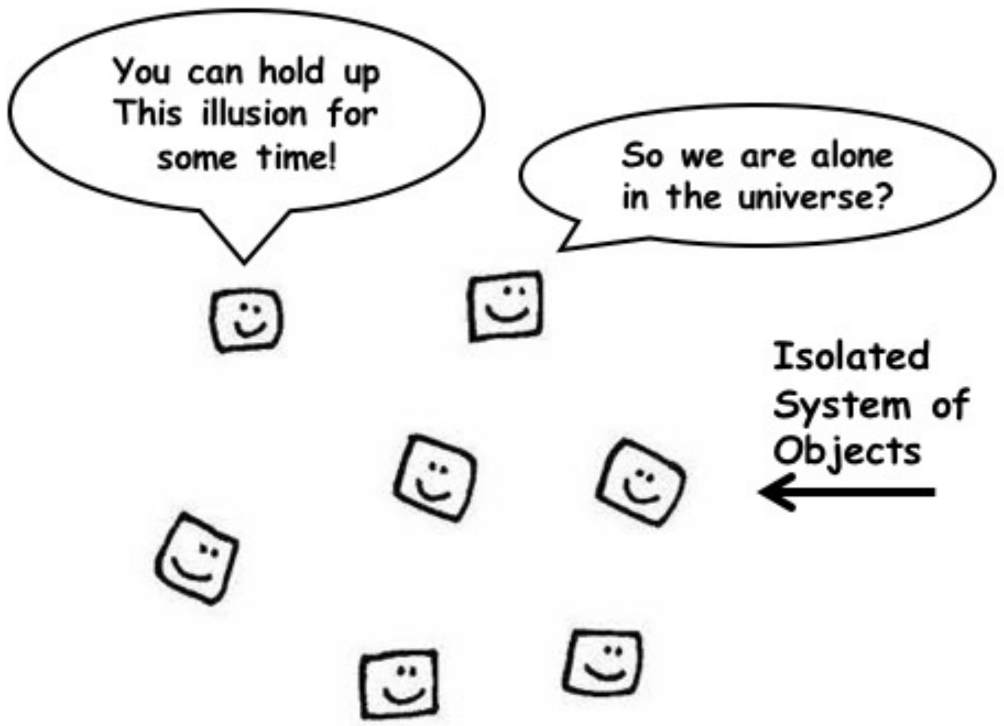




Conservation of Energy:

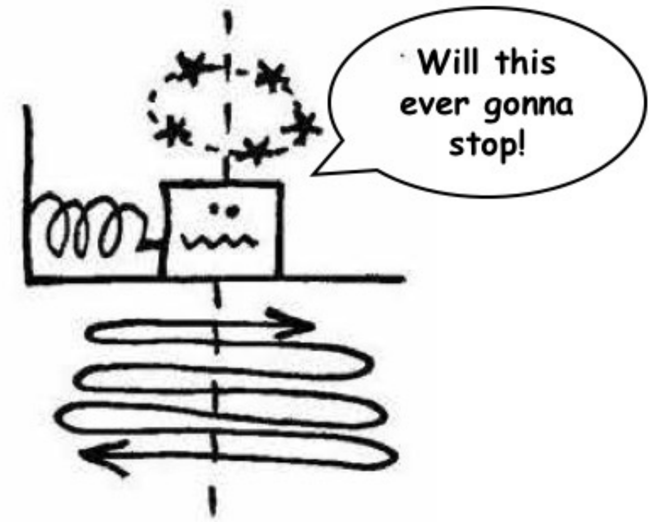
For an isolated system where the net external force on the system is zero and only internal conservative forces act, the total mechanical energy of the system remains constant. This is because the work done by any of the conservative forces within the system will transform its KE into PE (that is associated with that force) and vice versa. The total mechanical energy of a system is defined as the sum of all of the kinetic energies of the moving objects within the system plus the sum of all types of potential energies in the system, i.e.





When solving problems, it is important to identify the system and identify internal and external forces accordingly.

For example, if friction is neglected, the (block-spring) system can be considered to be an isolated system where only the internal conservative spring force act within it. This force will convert the KE of the block into the elastic PE of the (block-spring) system and back again into KE of the block continuously as it moves back and forth. However, the total mechanical energy of the (block-spring) system will remain constant at all times. Clearly the block does wish there is friction!



Similarly, if air resistance is neglected, then the only internal conservative force within the system of the (apple-Earth) is the force of gravity and the total mechanical energy of the (apple-Earth) system remains constant at all times

We can write this as

$$W = \Delta K$$

and since

$$W = -\Delta U$$

then

$$\Delta K = -\Delta U$$

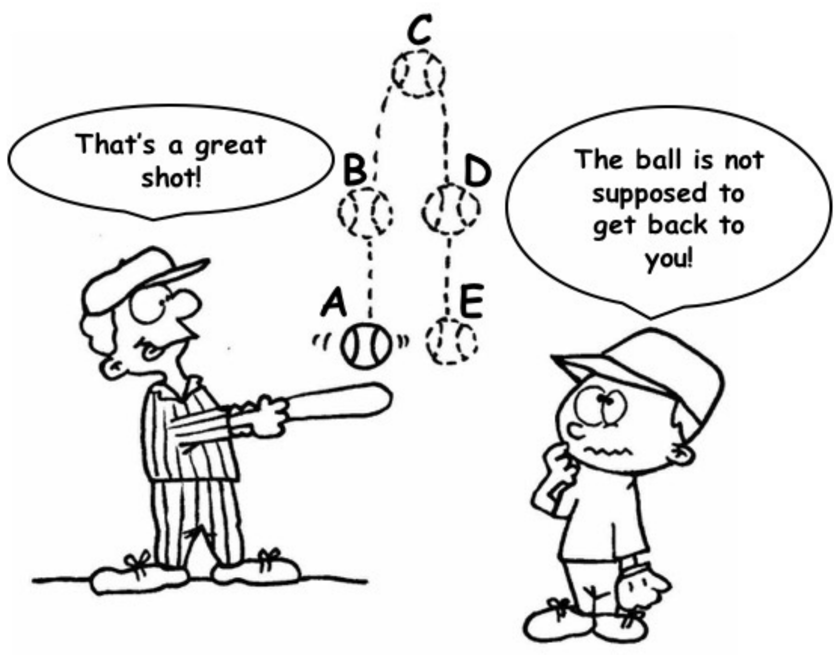
and

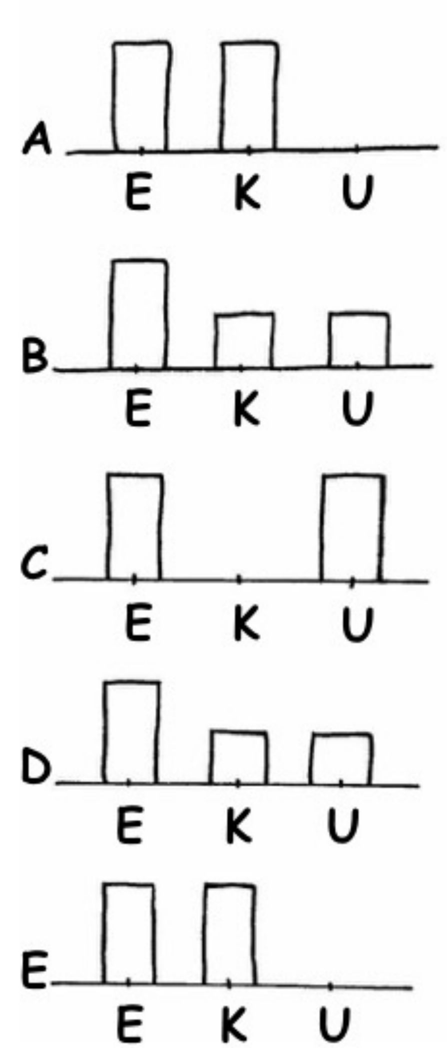
$$\Delta K + \Delta U = 0$$

or

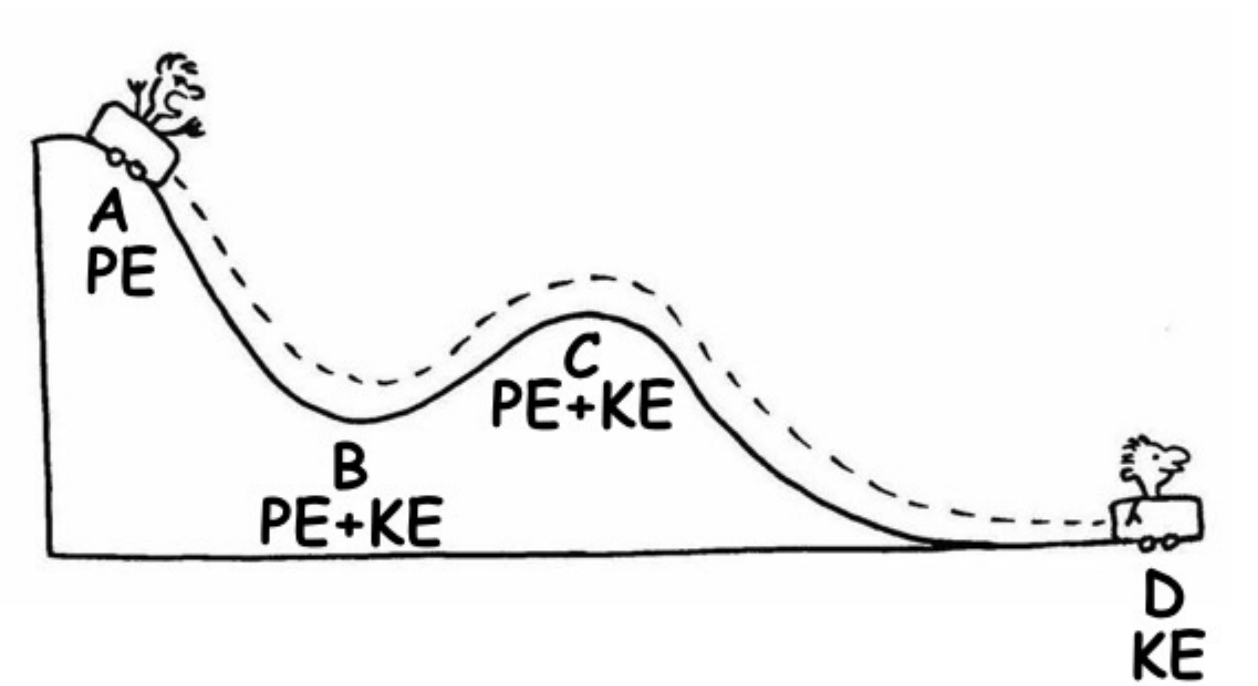
$$K_i + U_i = K_f + U_f$$

Consider a baseball thrown straight upwards. The figure below shows the KE and the gravitational PE of the (ball-Earth) system at each point and how the total mechanical energy E of the system remains constant at all times.

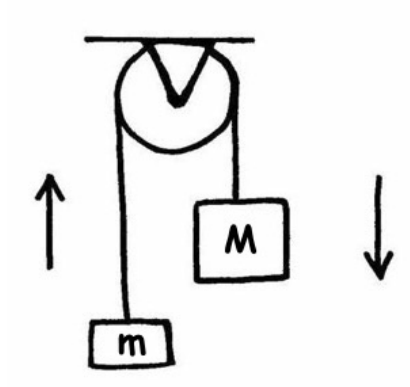




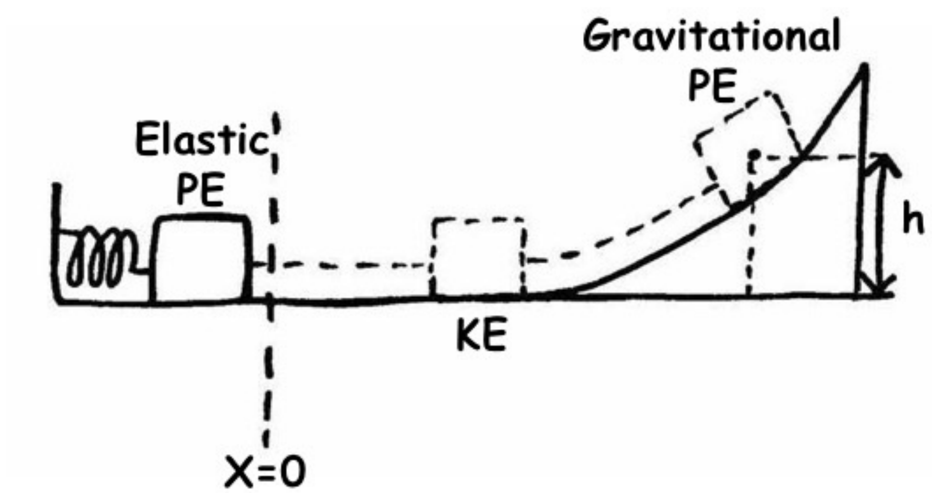
Another example is a roller coaster. If friction is neglected, then the (car-Earth) system can be considered to be isolated where the only internal force in the system is the force of gravity which is conservative. First it starts from rest with just a gravitational PE (at point A with much screams), then as it moves, some portion of the PE is converted into KE at different points (like at B and C), until it is totally KE at D. Using $K_i + U_i = K_f + U_f$, you may find the speed of the roller coaster at any point if you know its mass and the height in which it started from.



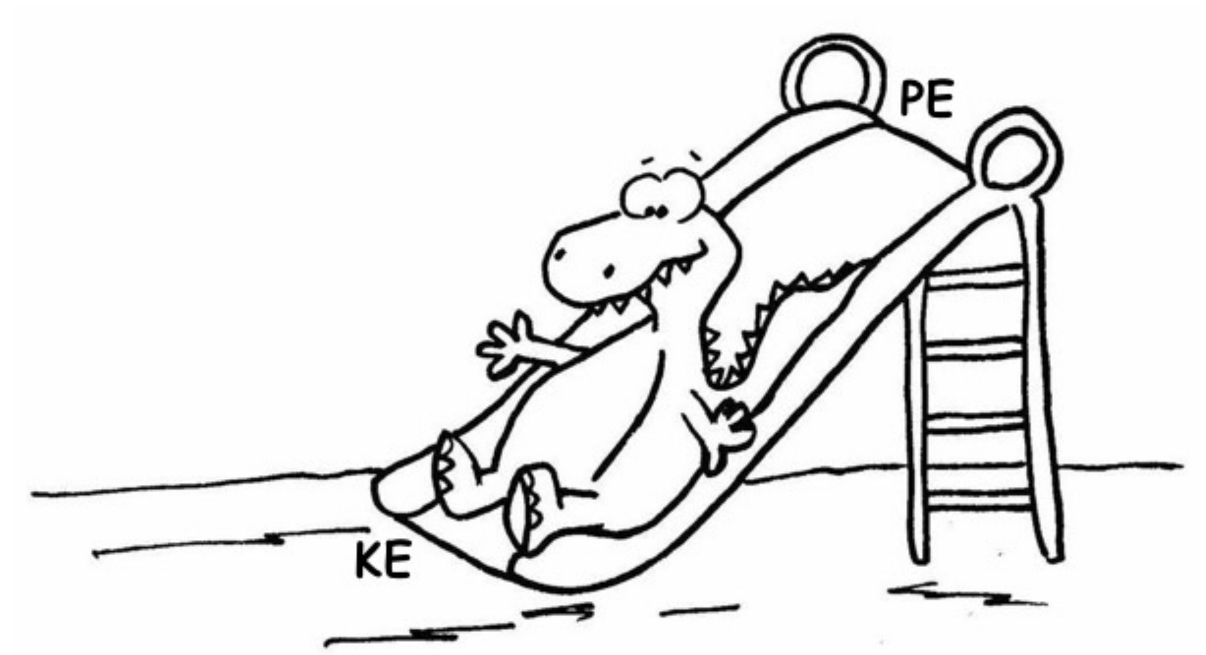
In an Atwood Machine, if friction in the rope and pulley is neglected, then the (blocks-Earth) system can be considered as an isolated system where gravity acts as the internal conservative force. Here, the gravitational PE of the heavier mass M as it moves downwards is converted into the KE of both m and M and also into the PE of m as it rises.



Another example is the system shown below, if friction is neglected, then the only forces acting on the (mass-spring-Earth) system is the forces of the spring and gravity, which are both conservative. The Elastic PE is converted into KE as the block is released and is then converted into the gravitational PE as it moves upwards along the ramp.

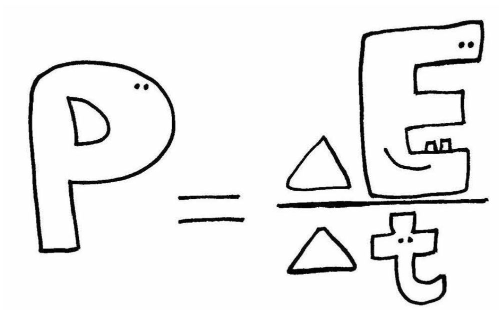


Note that the concept of energy can be used in solving problems without resorting to Newton's laws, especially when the exact form of the forces is complex and cannot be determined exactly. This dinosaur knows how to use PE for fun!



Power:

Power is the rate of energy transfer. In terms, of work, it is the rate at which work is done.



Its unit is joules per second (J/s) known as Watt (W).

$$1 \text{ W} = 1 \text{ J/s} = 1 \text{ kg.m}^2/\text{s}^3$$

Another unit used is horsepower (hp)

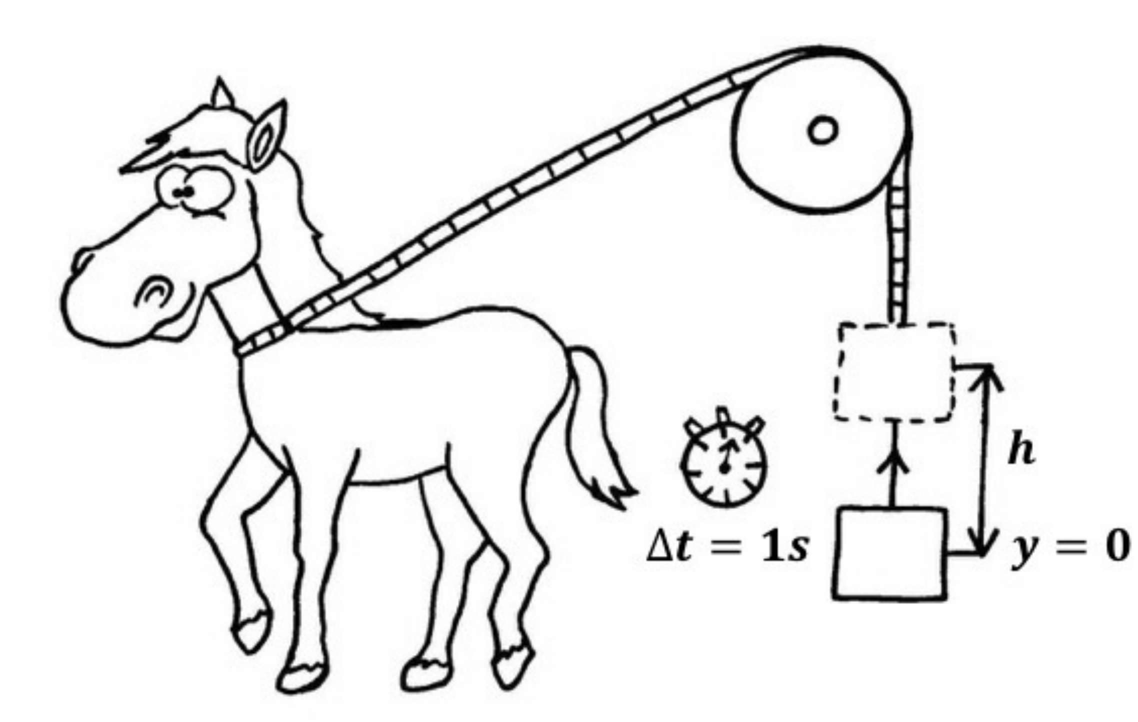
$$1 \text{ hp} = 746\text{W}$$

For an approximate calculation, suppose a horse is lifting a block of mass 75 kg a distance of 1 m in 1 s. Taking $y = 0$ at the Earth's surface, the work done by gravity is

$$W = -\Delta U = -mgy_f = -mgh$$

If the block is lifted at a constant speed, then

$$\Delta K = 0$$



and therefore, the total work done on the block is

$$W_{ext} = W_{horse} + W_{gravity} = 0$$

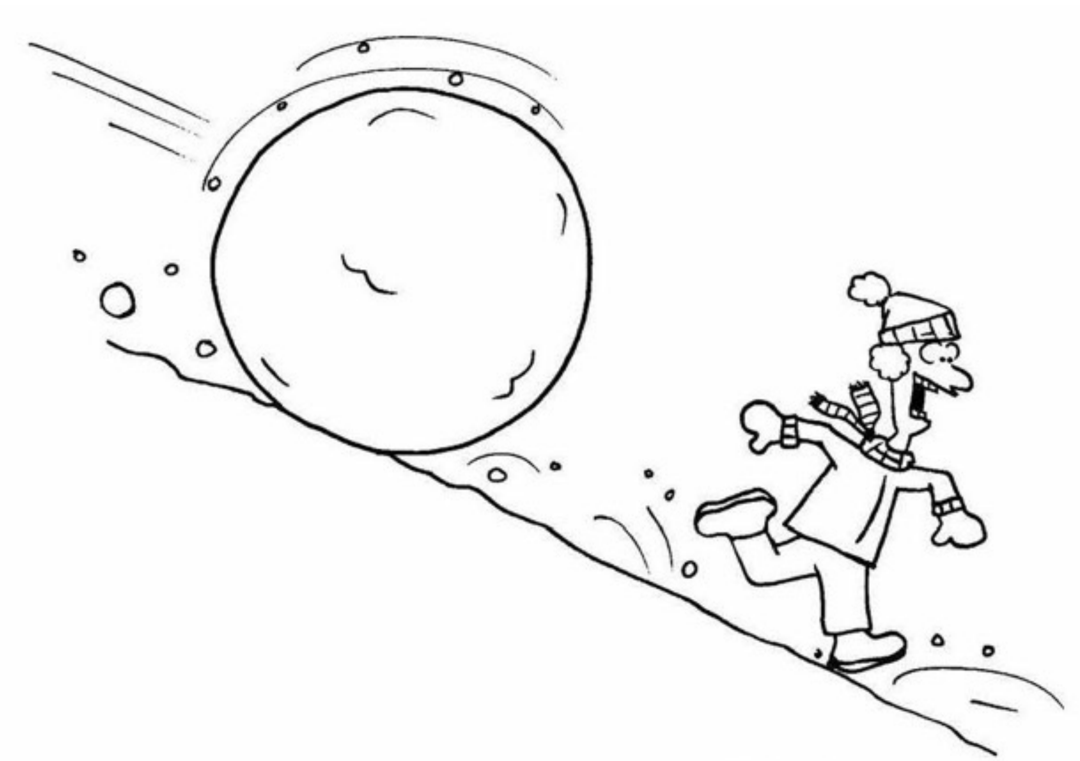
here the block-Earth system is not isolated since the horse is exerting a force on the block. The work done by the horse is then

$$W_{horse} = +mgh$$

which is the amount of energy transferred to the block. If the block was lifted in one second, then the power delivered to the block is one horsepower

$$P = \frac{mgh}{\Delta t} = \frac{(75 \text{ kg})(10 \text{ m/s}^2)(1\text{m})}{(1\text{s})} = 750 \text{ W} = 1 \text{ hp}$$

Linear Momentum and Collisions

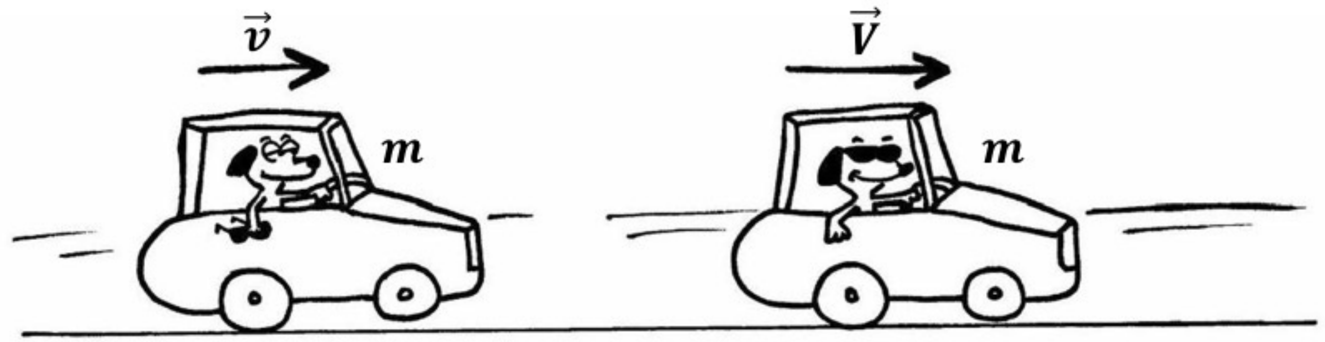


Linear Momentum and Collisions:

The Linear momentum of a particle is a vector quantity defined as

$$\vec{p} = m\vec{v}$$

Its SI units is kg.m/s. For example, if there are two identical cars, the faster car has more momentum.

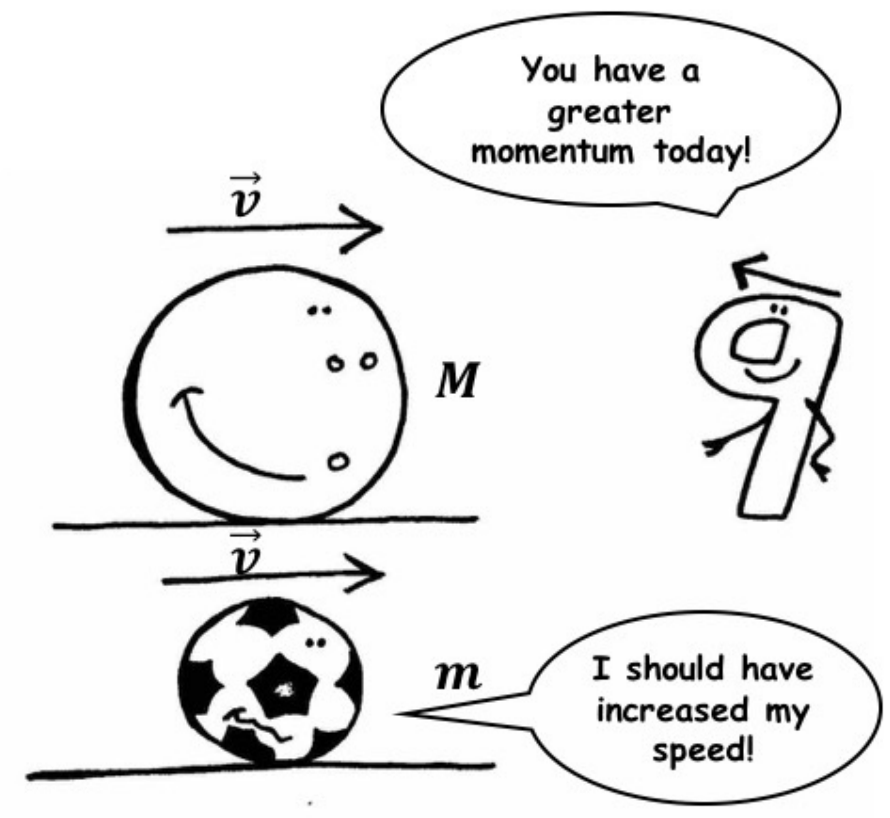


Also, if a bowling ball and a football have the same speed, then the bowling ball has more momentum. Newton's second law in terms of momentum can be written as

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Since if the mass is constant, this expression reduces to

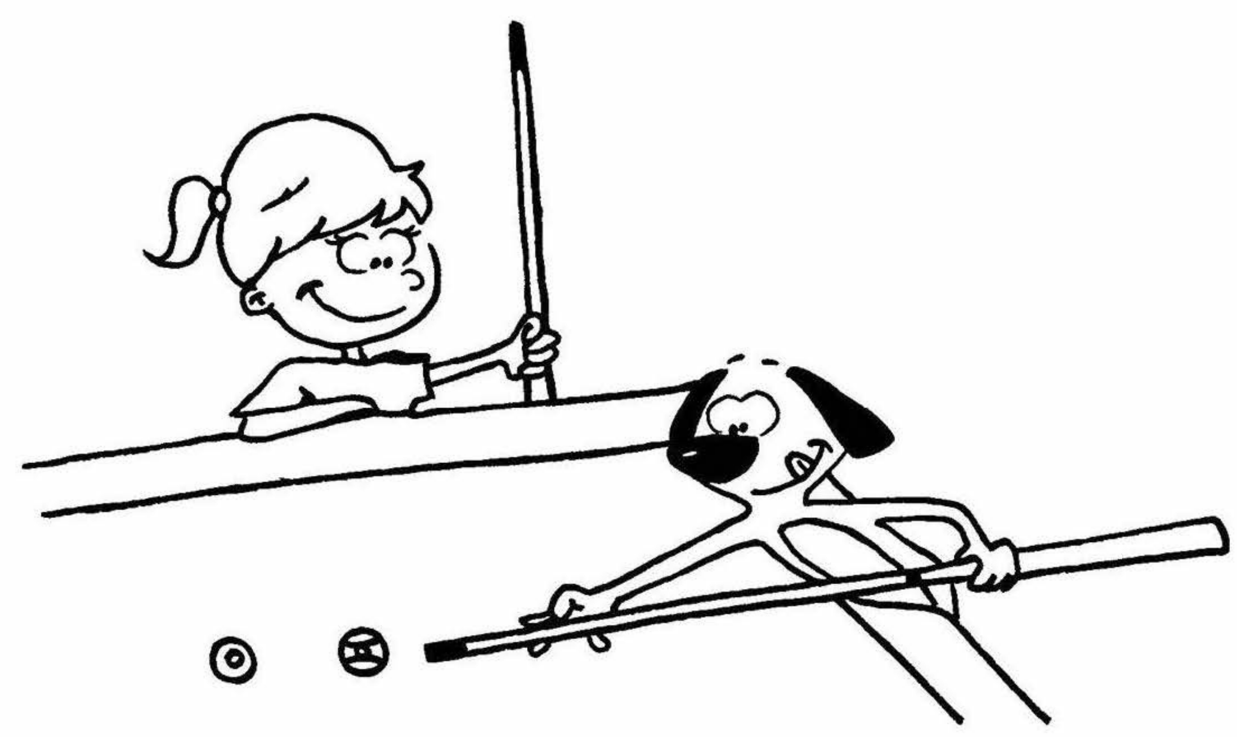
$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$



I.e. the net force acting on a particle (or particle-like object) is equal to the time rate of change of its momentum. This is the general form of Newton's second law that is valid even when the mass of the object is changing over time such as a rolling snow ball that picks up snow and grows as it moves or a rocket ejecting fuel. Another example is the increase of the mass of an object as it approaches the speed of light according to Einstein's special theory of relativity.

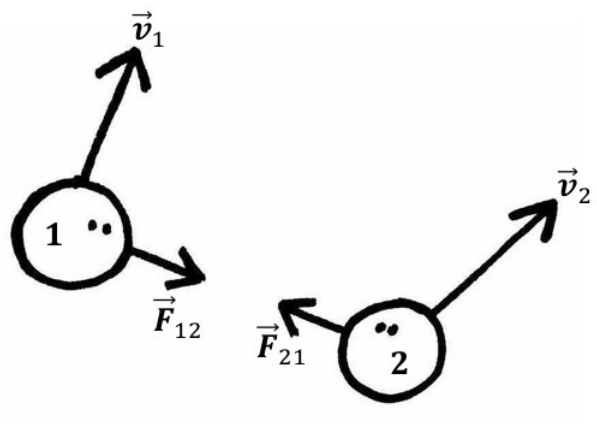
The concept of linear momentum is especially useful in treating collisions without the need to use Newton's second

law and know the exact form of the forces involved. For example, consider two bodies colliding with each other. During the collision, each body will exert a force on the other known as the impulsive force that acts over a very short period of time and is usually very large. Such forces are complex functions of time and therefore it is difficult to use Newton's second law to solve collision problems. We will see how the law of conservation of linear momentum is applied immediately before and immediately after the collision without the need to know the exact form of the impulsive forces. The collision between billiard balls is an example of an elastic collision as we shall see later.



Conservation of Linear Momentum:

Suppose we have a system consisting of two particles where the net external force acting on the system is zero (the system is isolated), and if there isn't any exchange of matter with the surrounding (the system is closed), then the total linear momentum of the system remains constant (conserved) at all times. This can be found from



$$\sum \vec{F} = \frac{d\vec{p}_{tot}}{dt} = 0$$

and since

$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2$$

where $\vec{p}_1$ and $\vec{p}_2$ are the linear momentums of particle 1 and 2 respectively, then we have

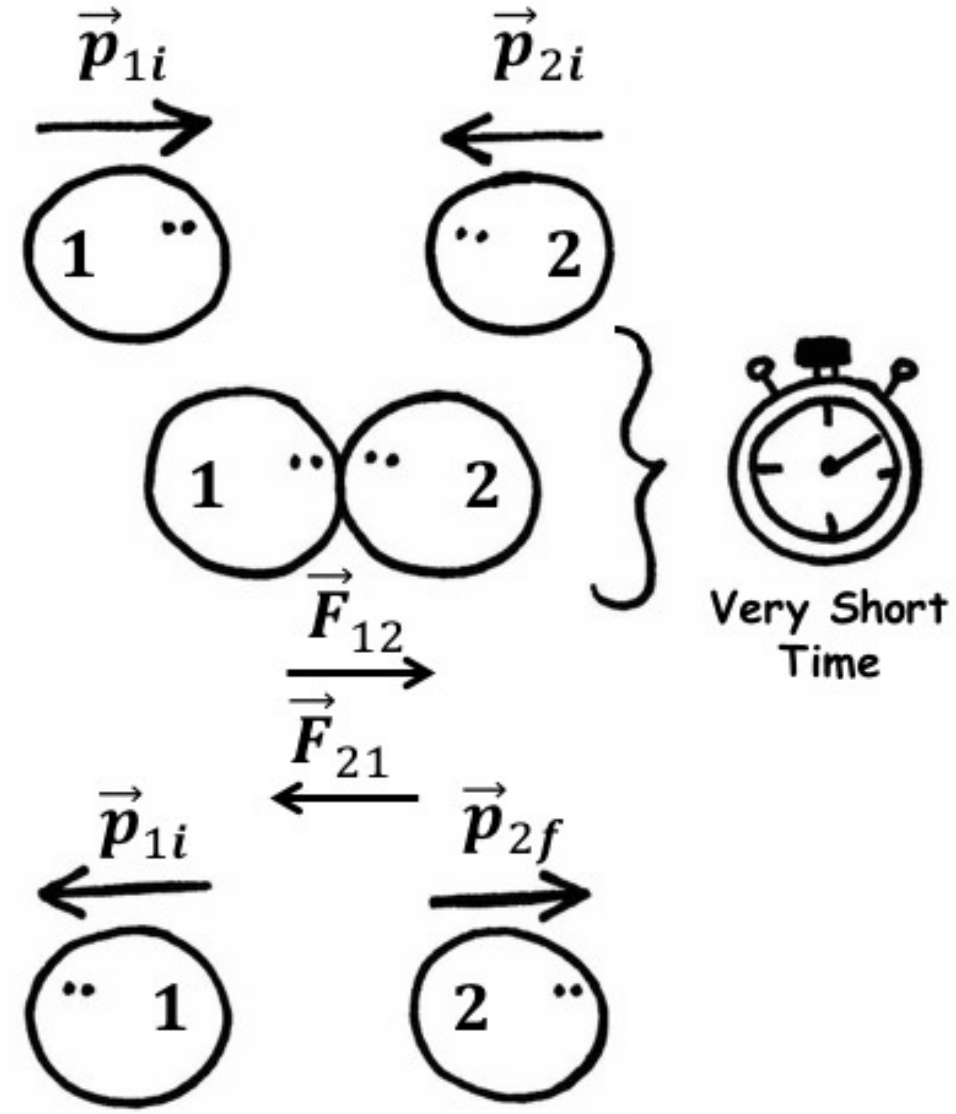
$$\frac{d\vec{p}_{tot}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt}$$

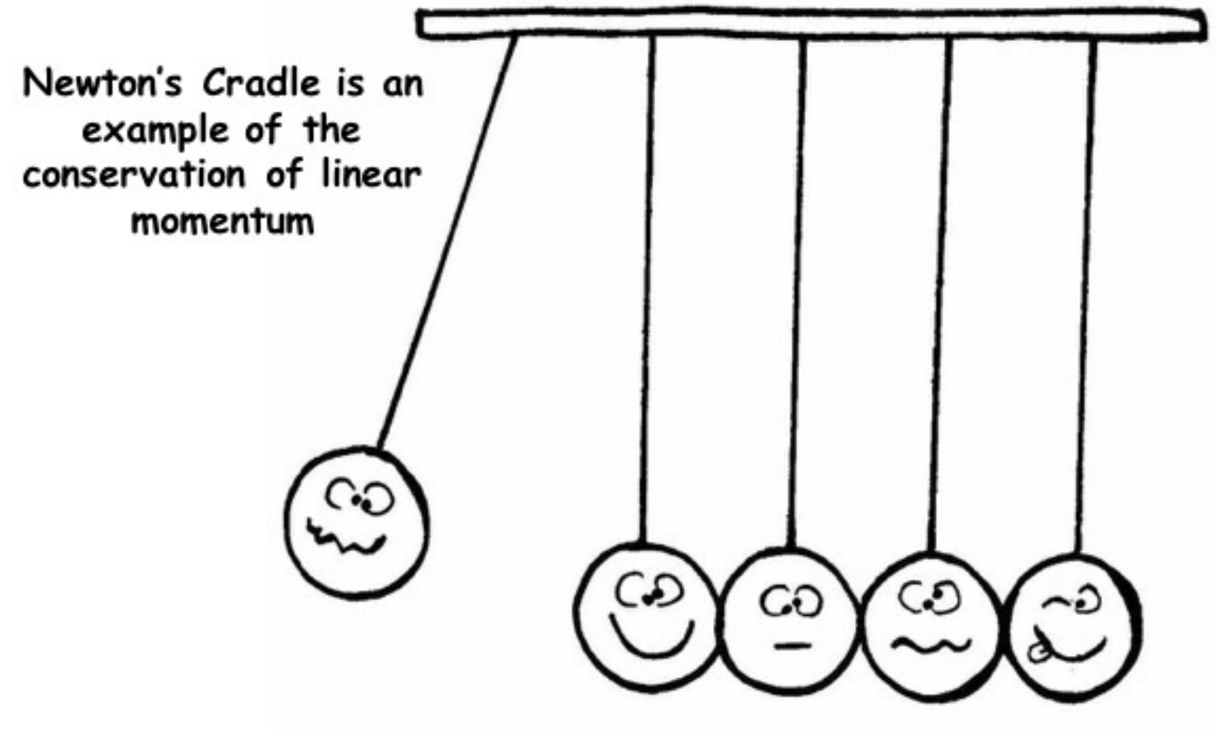
$$= \vec{F}_{12} + \vec{F}_{21} = \vec{F}_{12} - \vec{F}_{12} = 0$$

$$\Rightarrow \vec{p}_{tot} = \text{constant}$$

$$\vec{p}_{(tot)i} = \vec{p}_{(tot)f}$$

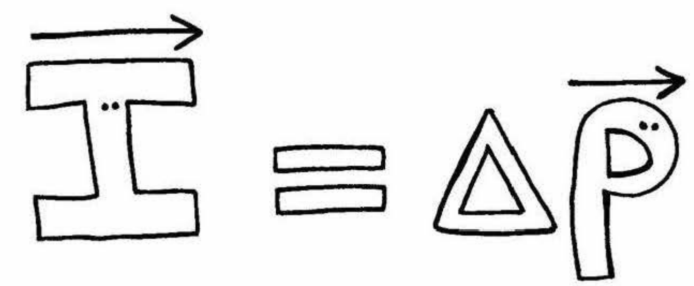
Where $\vec{F}_{21}$ is the force exerted on particle 2 by particle 1 and $\vec{F}_{12}$ is the force exerted on particle 1 by 2, which using Newton's third law we know that $\vec{F}_{21} = -\vec{F}_{12}$. I.e. the linear momentum of each particle may change as they interact with each other, but the total linear momentum of the system will remain constant at all times. This situation is similar to what happens in a collision since during the very short time of a collision, the system of the two colliding objects can be considered to be isolated. This is because the impulsive force that each object exerts on the other is very large such that any other forces present during that short time (like friction or gravity) may be neglected. In this case $\vec{p}_i$ and $\vec{p}_f$ are the linear momenta immediately before and immediately after the collision.





Impulse:

Impulse is a vector quantity that defines how a force acting on a particle changes the linear momentum of that particle and is given by

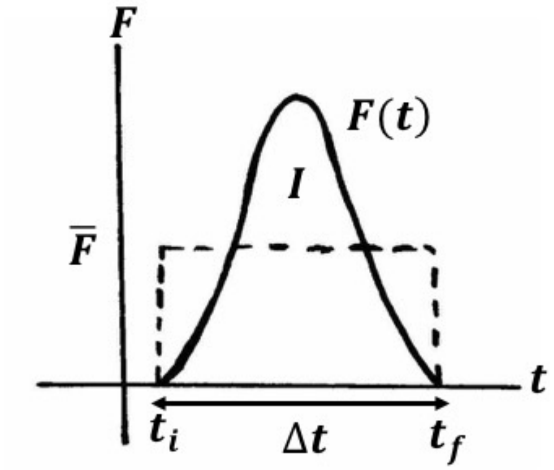


$$\vec{I} = \Delta\vec{p} = \int_{t_i}^{t_f} \vec{F}\,dt$$

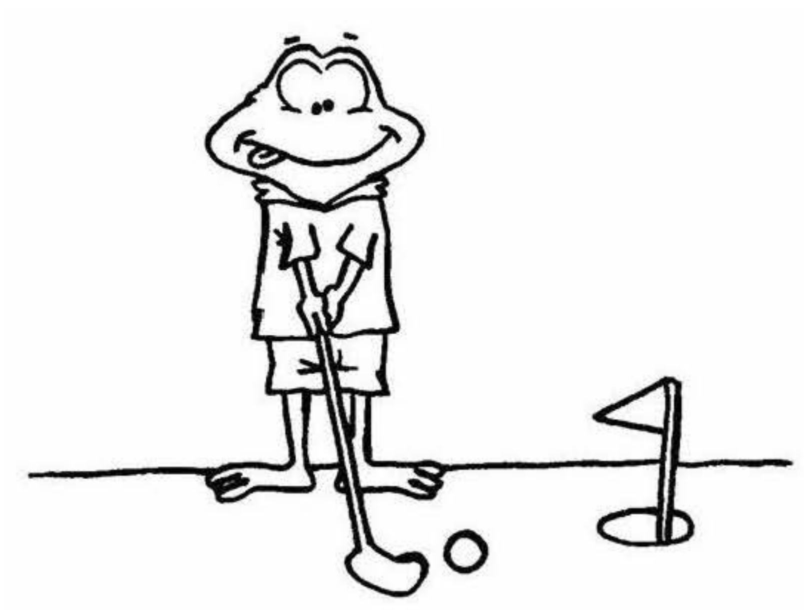
$$\vec{F}_{avg} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{F}\,dt$$

$$\vec{I} = \vec{F}_{avg}\Delta t$$

In the case of a collision, the impulsive force that each particle exerts on the other during the collision time can take the form shown in the graph and can be replaced by $\vec{F}_{avg}$ which gives the same impulse as $\vec{F}(t)$.



Consider our friend frog who regularly plays golf to relax in his time away from the pond.

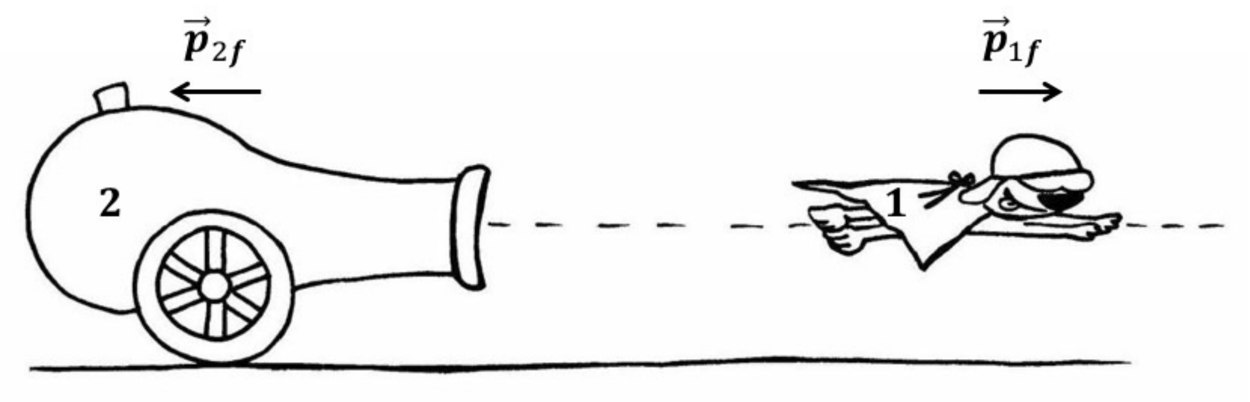


As he hits the ball, the force exerted by the club on the ball will vary from zero when contact is made to a maximum and back to zero when the ball leaves the club in a manner similar to the bell-shaped curve shown above. The impulse given to the ball is $I = \Delta p = mv_f - 0 = mv_f$ since it starts from rest. If the ball is given a speed of $v_f = 45\ \mathrm{m/s}$ and if its mass is $m = 50\times10^{-3}\mathrm{kg}$, then $I = 2.25\ \mathrm{kg.m/s}$. If the time of contact with the club is $\Delta t = 4.5\times10^{-4}\mathrm{s}$, then using $I = F_{avg}\Delta t$, the average force exerted by the club on the ball is $F_{avg} = 5000\ \mathrm{N}$, which is extremely large compared to the weight of the ball, which is only 0.49N. This is why it is possible to consider the two colliding objects as an isolated system where only the internal impulsive forces act during the collision time.

Such a force can easily break a glass when it falls on the ground and that is exactly what this cat did when it didn't like the juice it was served!

To minimize the force of impact, the trampoline increases the collision time which reduces that force

An example of the conservation of linear momentum is the recoil speed of a canon when firing a ball, or in this case Bud, who wanted to fly as a superdog.



Since the net force along the x-direction is zero for the (Bud-canon) system, the total linear momentum is conserved along that direction and so we have

$$\vec{p}_{(tot)i} = \vec{p}_{(tot)f}$$

or

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

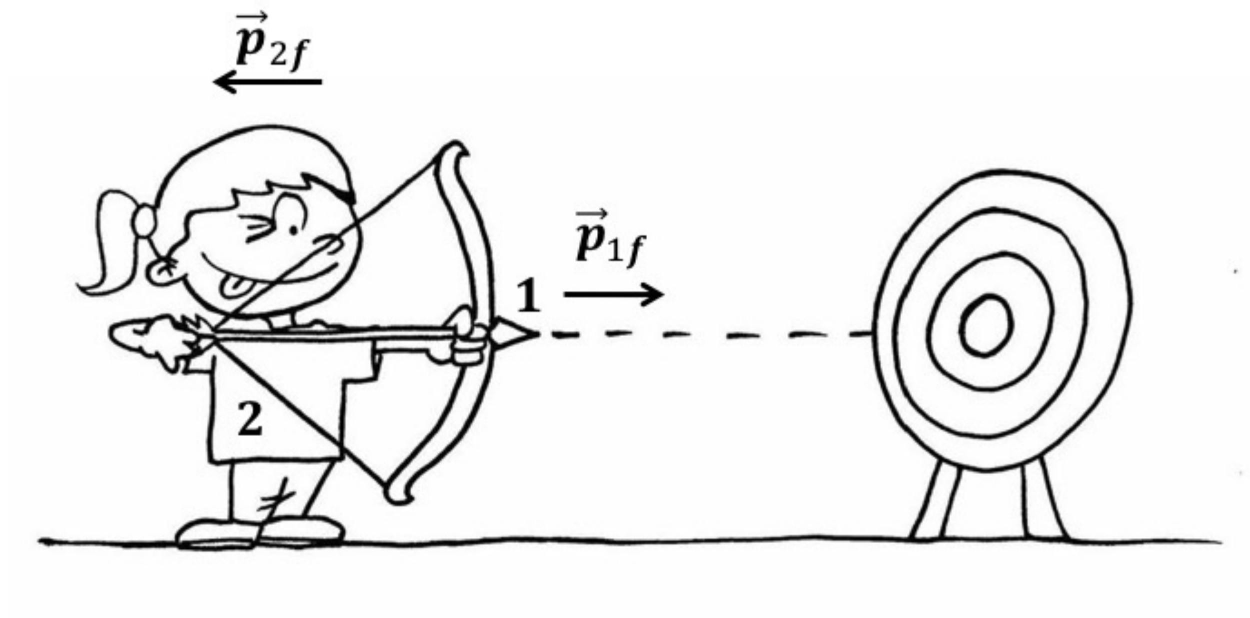
The canon (1) and Bud (2) were both at rest before the firing, therefore

$$\vec{p}_{1f} + \vec{p}_{2f} = 0$$

$$\vec{p}_{2f} = -\vec{p}_{1f}$$

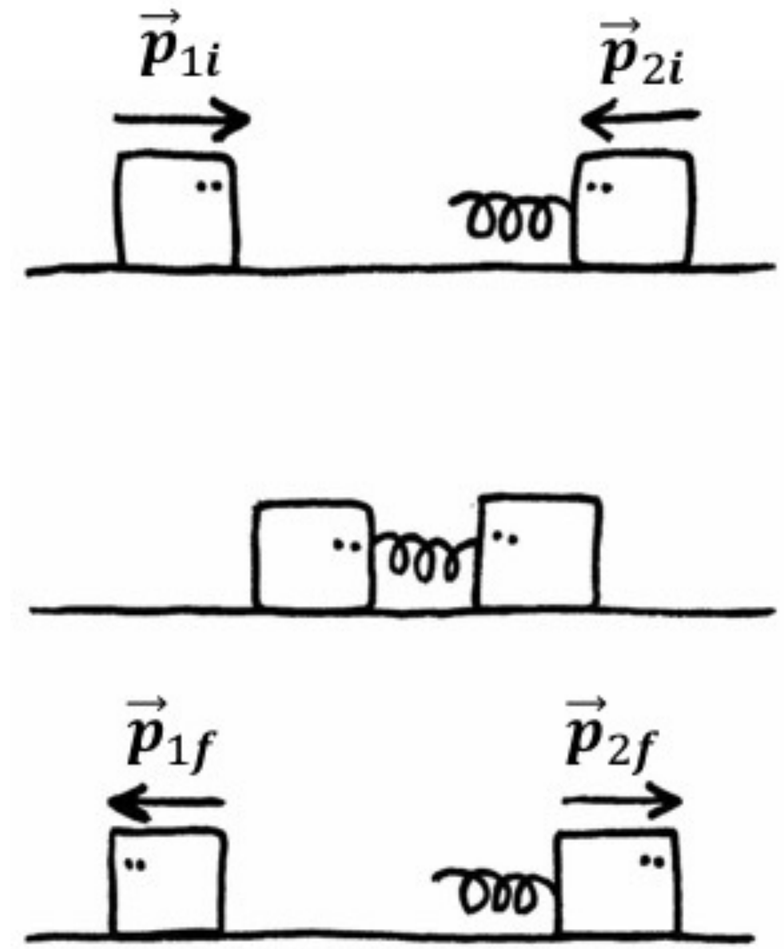
$$\vec{v}_{2f} = -\frac{m_1}{m_2}\vec{v}_{1f}$$

which is the recoil speed of the canon. Another example is when Sophie shoots an arrow. If the surface she is standing on is ice, we may ignore friction and assume that the net external force acting on the (Sophie-arrow-bow) system along the x direction is zero and therefore the total linear momentum is conserved in that direction and the recoil speed of Sophie including the bow is $\vec{v}_{2f} = -\frac{m_1}{m_2}\vec{v}_{1f}$ as in the case for the canon.



Elastic and Inelastic Collisions:

If the impulsive forces between the objects acting during the time of the collision are conservative, such as the spring force shown below, then the total kinetic energy of the system as well as its total linear momentum are both conserved and the collision is said to be elastic. On the other hand, if the impulsive forces are non-conservative, the collision is an inelastic collision and only the total linear momentum is conserved, while some of the total KE of the system is lost and converted into heat.



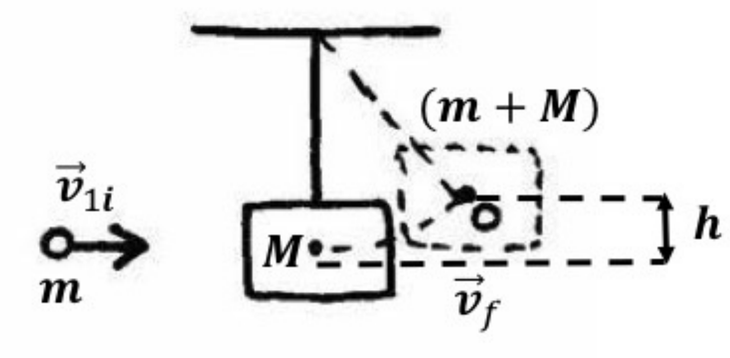
In a perfectly inelastic collision the two objects stick together.

The ballistic pendulum shown below is a system used to measure the speed of a bullet where it is embedded in the block in a perfectly inelastic collision and where both swing to a height h. The speed of the (block + bullet) combined system just after the collision can be found from the conservation of linear momentum in the x-direction

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$m\,\vec{v}_{1i} + 0 = (m + M)\vec{v}_f$$

$$\vec{v}_f = \frac{m\vec{v}_{1i}}{(m + M)}$$



This speed determines the initial KE of the (block + bullet) that will be transformed into PE as the system reaches the maximum height h. From the conservation of energy, h is related to v_f through

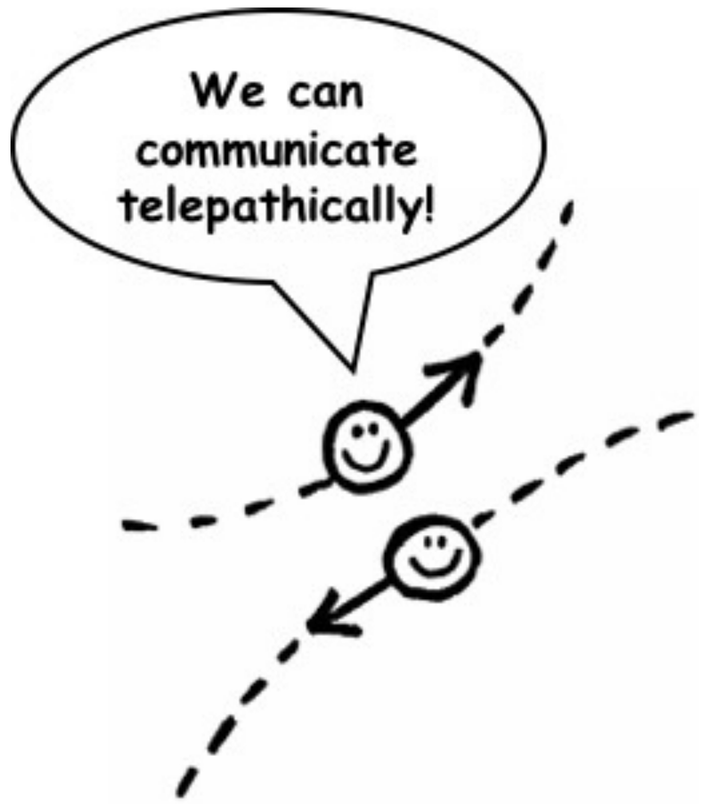
$$v_f = \sqrt{2gh}$$

or in terms of the speed of the bullet

$$v_{1i} = \frac{(m + M)}{m}\sqrt{2gh}$$

Therefore, it is possible to find the initial speed of the bullet by measuring h.

One thing to note is that a collision may not always entail physical contact, such as at the atomic scale, for example between a proton and an alpha particle, where since both are positively charged they will repel and never come into physical contact in a collision.



Torque:

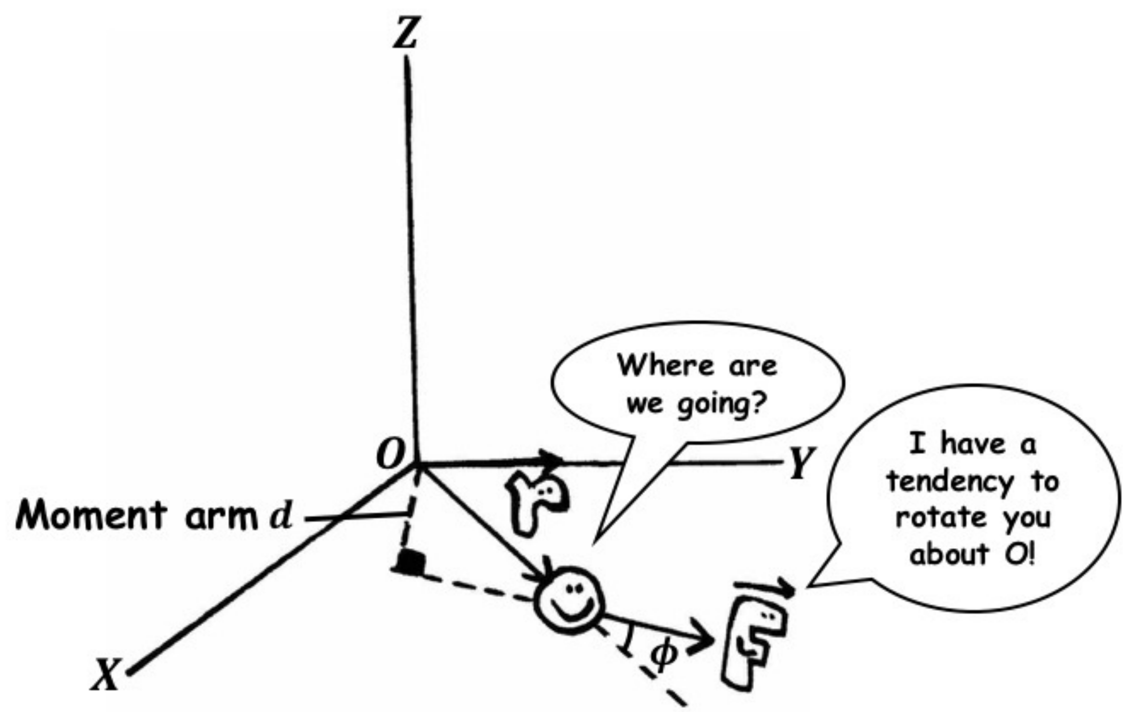
Torque is a vector quantity that measures the tendency of a force to rotate an object about an axis through O as shown, and it is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

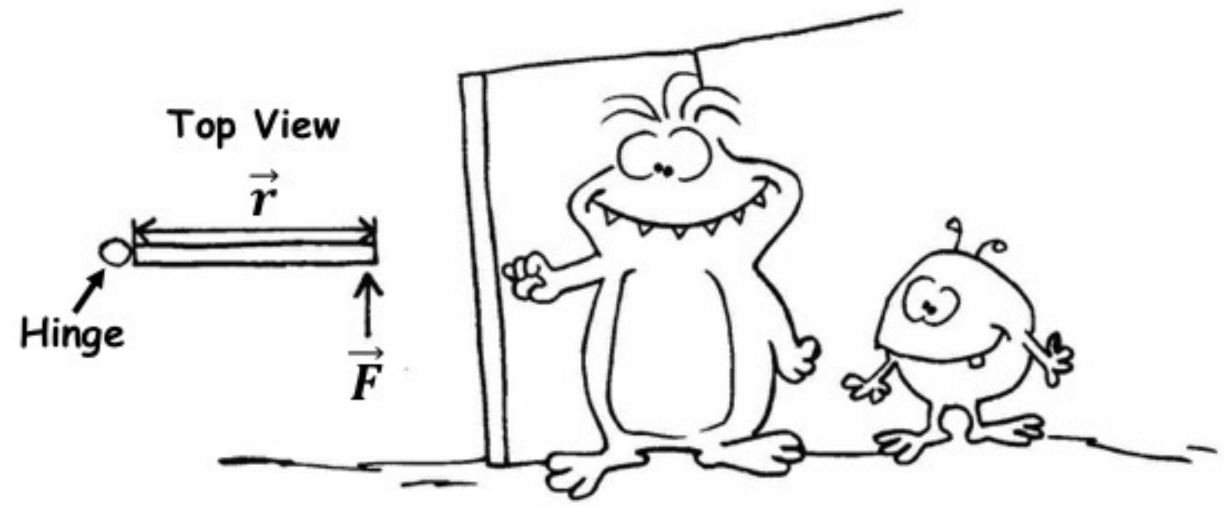
where $\vec{r}$ is the vector position of the object relative to O. The magnitude of the torque τ is

$$\tau = rF\sin\phi$$

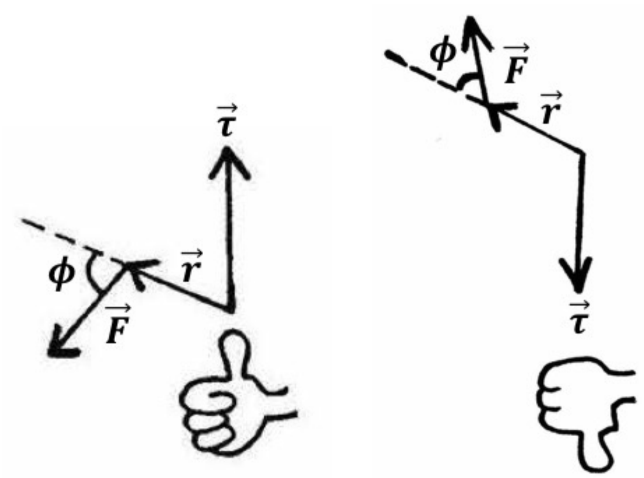
where ϕ is the small angle between $\vec{r}$ and $\vec{F}$ and $d = r\sin\phi$ is the moment arm of $\vec{F}$ as shown in the figure below. Therefore, increasing the moment arm will increase the torque.



For example, when you open a door, it is easier to open at the knob than at a position closer to the hinge since that will increase the torque.

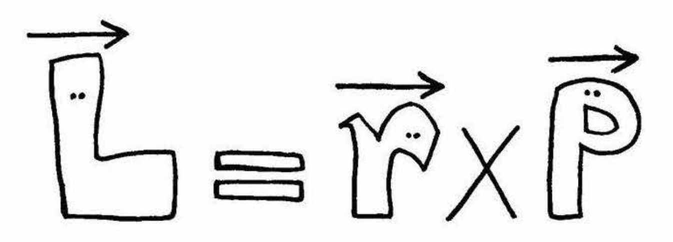


The SI unit of torque is N.m. To find the direction of the toque $\vec{\tau}$, curl your right-hand fingers from $\vec{r}$ to $\vec{F}$ as shown, the direction of your thumb will be in the direction of $\vec{\tau}$. Also from the definition of the vector product, $\vec{\tau}$ is always perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$. As you can see, in the first case the object will rotate counter clockwise and $\vec{\tau}$ is directed upwards, while in the second case it will rotate clockwise and $\vec{\tau}$ is directed downwards.



Angular Momentum:

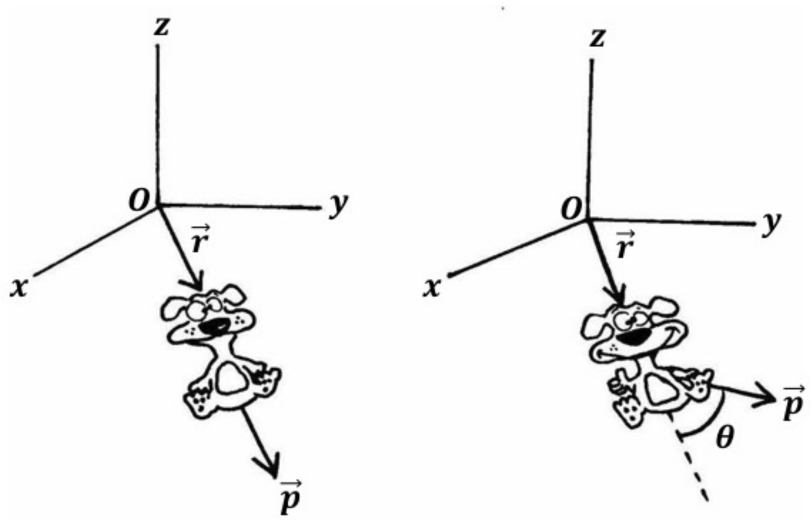
For a particle of mass m and linear momentum $\vec{p}$, its angular momentum is defined as



where $\vec{r}$ is the vector position of the object relative to O. The SI unit of angular momentum is $kg.m^2/s$. The magnitude of the angular momentum is

$$L = rp\sin\phi$$

where ϕ is the small angle between $\vec{r}$ and $\vec{p}$. Consider Bud (who can be considered as a particle here), in the first case when the angle between $\vec{r}$ and $\vec{p}$ is zero, the angular momentum of Bud about O is zero. In the second case $L \neq 0$ since $\phi \neq 0$.



Newton's Second Law in Angular Form:

The net torque acting on a particle is equal to the change in the angular momentum of that particle, i.e.,

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum:

If the net external torque acting on a system is zero, then the total angular momentum of the system remains constant (conserved), i.e.,

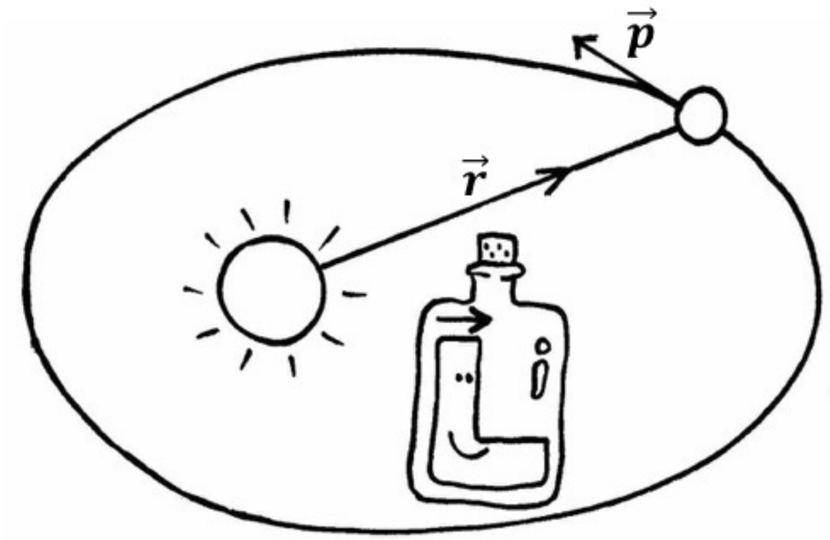
If

$$\sum \vec{\tau} = 0$$

then

$$\vec{L}_{tot} = 0$$

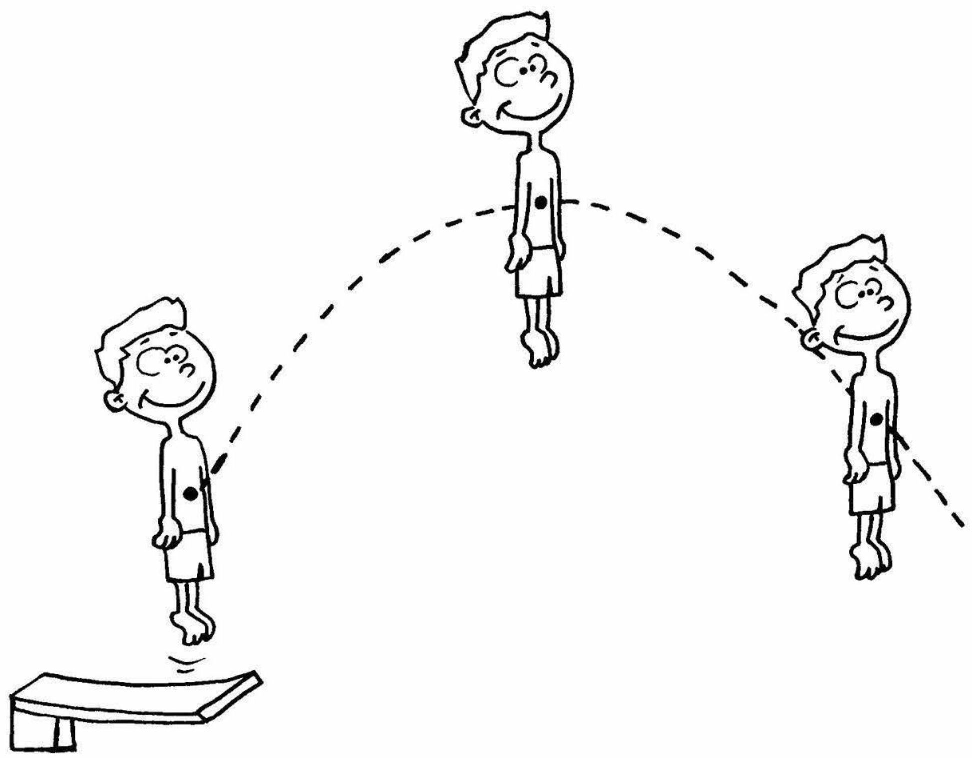
$$\Rightarrow \vec{L}_i = \vec{L}_f$$



An example is a planet orbiting the sun. Since the gravitational force between the sun and the planet exerts zero torque, the angular momentum of the planet is conserved.

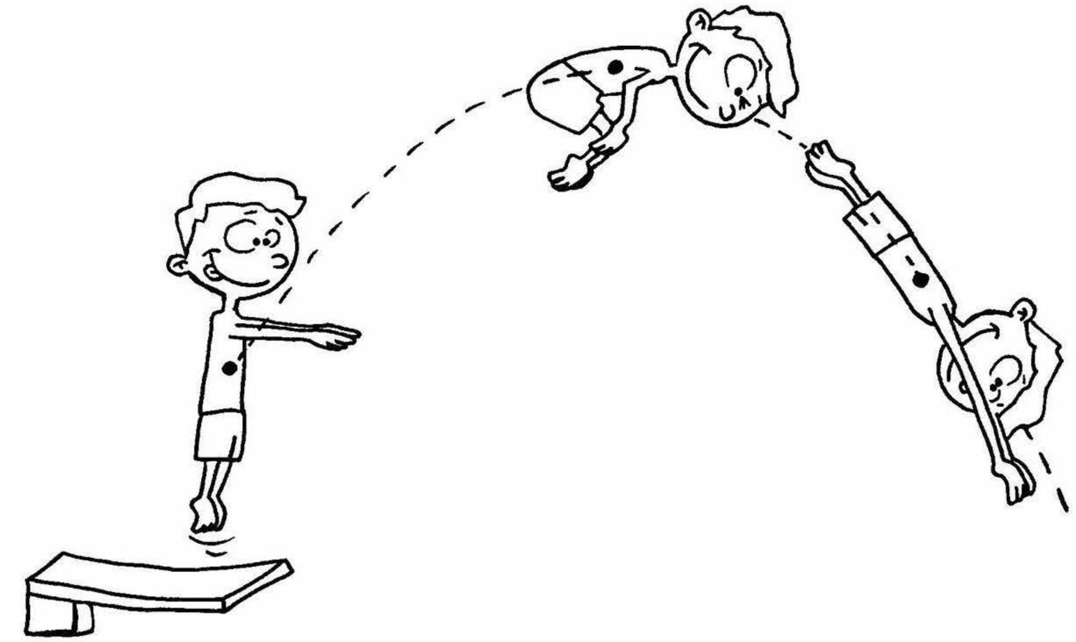
Center of Mass:

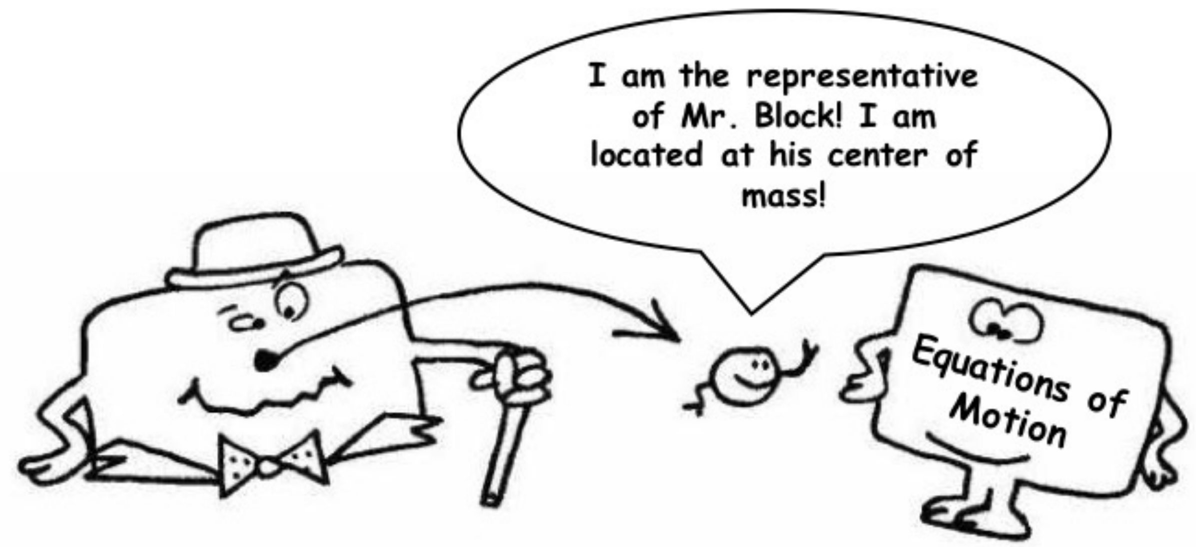
As we have discussed earlier, an object can be treated as a particle when all of its parts move in the same way, for example if you jump in a pool while freezing your body.



If different parts of the object move in different ways while it moves, such as when doing a summersault, then the object must be treated as a system of particles and its motion is represented by the motion of its center of mass. The point of the center of mass behaves as if all of the mass of the object is located there and as if the net external force is applied there.

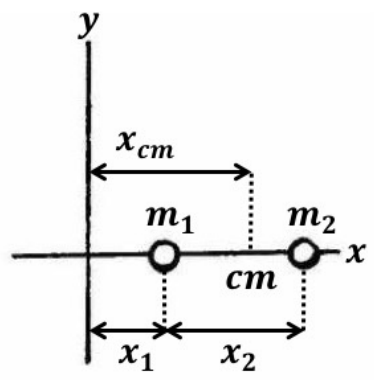
As you can see, the center of mass of the boy follows a parabolic path just like a particle in a projectile motion.





Center of mass of two particles:

Consider two particles of masses m_1 and m_2, the center of mass of the system is then located at the position

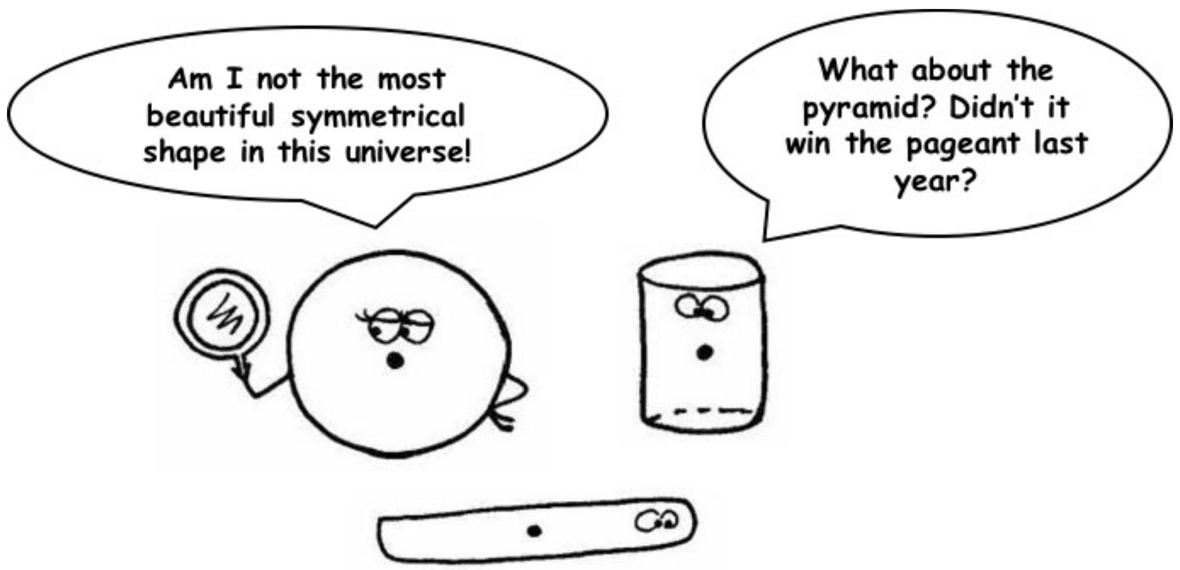


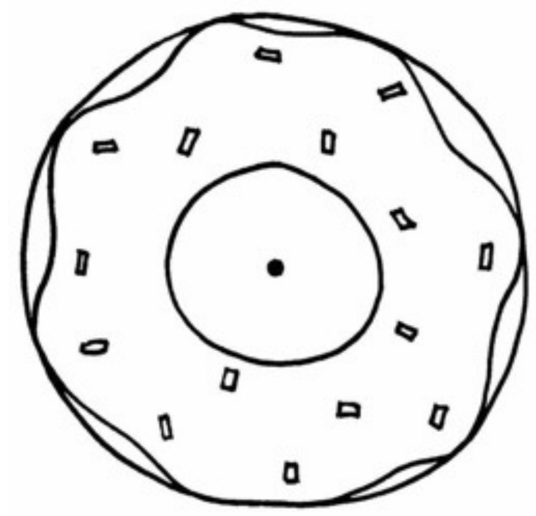
$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

For an extended 3D object, the center of mass position is given by

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm$$

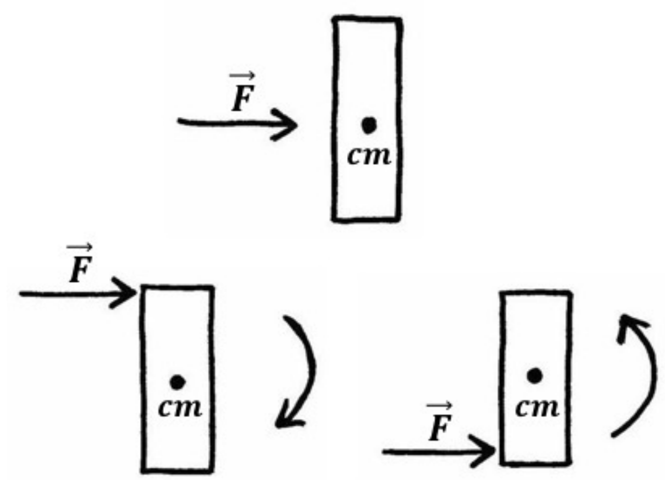
In the case of a homogenous symmetric object, the center of mass is located at its geometric center.





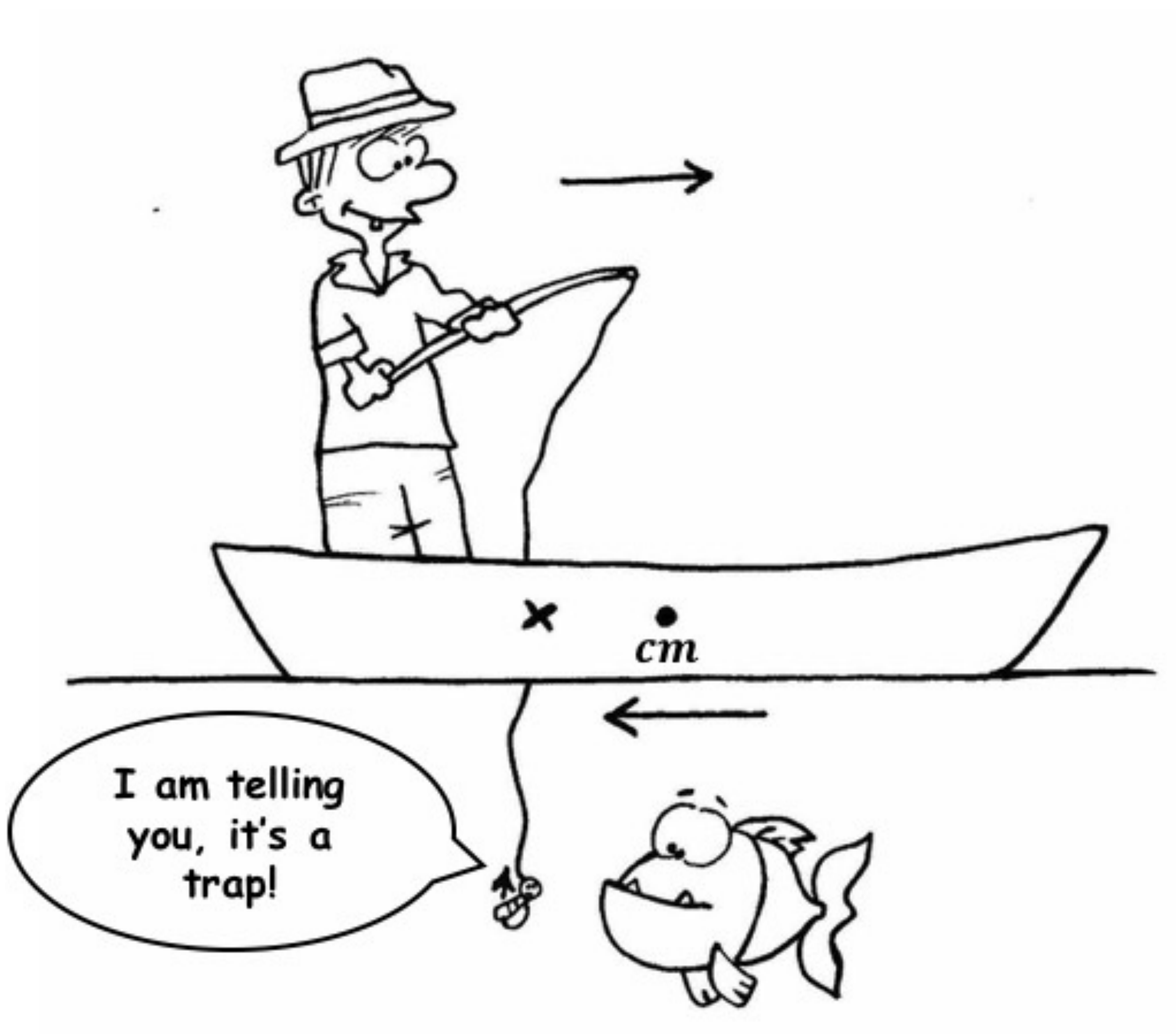
The point of the center of mass need not to be located within the object such as the center of mass of a donut.

Suppose a force $\vec{F}$ is applied to a uniform rod and its line of action passes through the center of mass, then the rod will behave as a particle located at the center of mass and it will accelerate in the direction of that force.



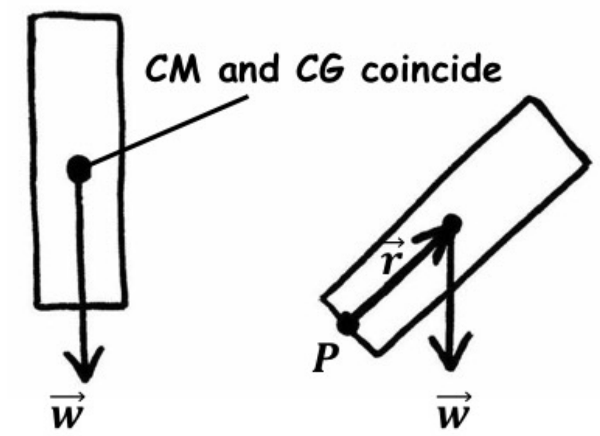
If the line of action of $\vec{F}$ does not pass through the center of mass, the whole rod will not move forward, instead it will rotate about the center of mass. This is due to the force generating a torque $\vec{\tau}$ about the center of mass.

A fisherman is standing on a boat, by neglecting air and water resistance, the net force acting on the center of mass of the (fisherman+boat) is zero. This means that if the fisherman moves to the left, the boat will have to move to the right in such a way as to keep the center of mass of the (fisherman+boat) system at the same position.



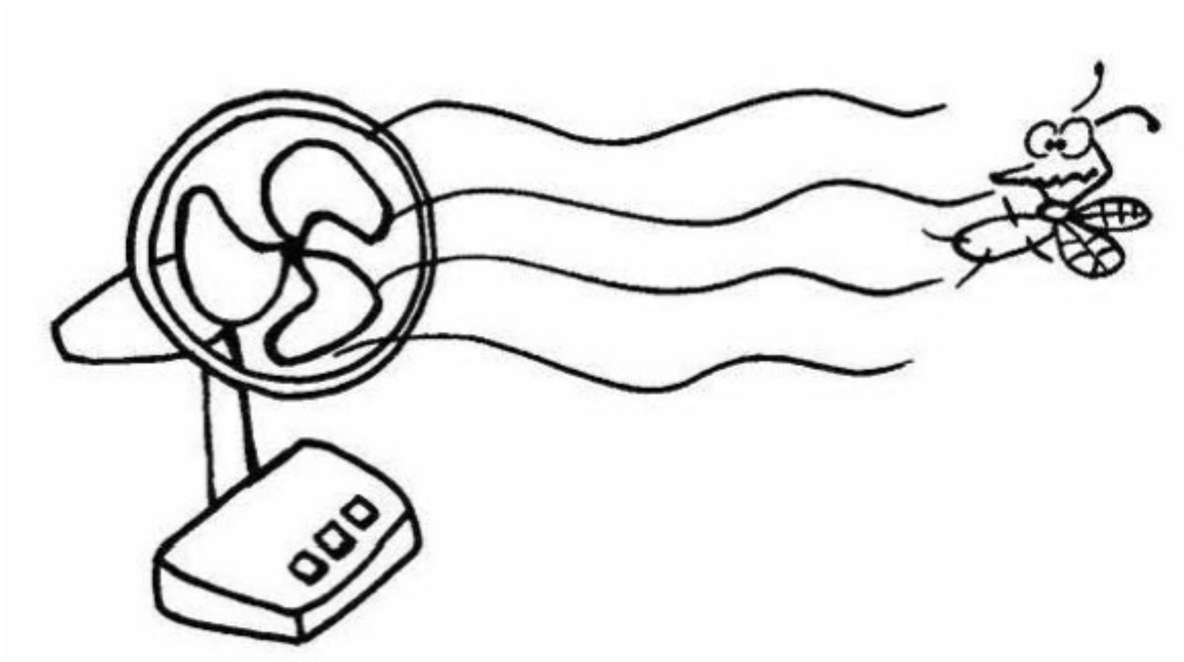
Center of Gravity (CG):

The center of gravity is the point at which the weight force $\vec{w}$ acts on the body (the weight is the resultant gravitational force on the body which is the vector sum of all of the gravitational forces on the individual masses forming the body). If the body is in a uniform gravitational field (more on that later), the center of gravity will coincide with the center of mass (CM). For example, for the rod shown, the weight force acts at the same position of the center of mass. If the rod is titled such that the line of action of its weight does not pass through its base, the weight will produce a torque around the point P which will rotate the rod about P resulting in it falling to the ground. Otherwise if the line of action of the weight passes through its base it will keep standing or fall back to the standing position.



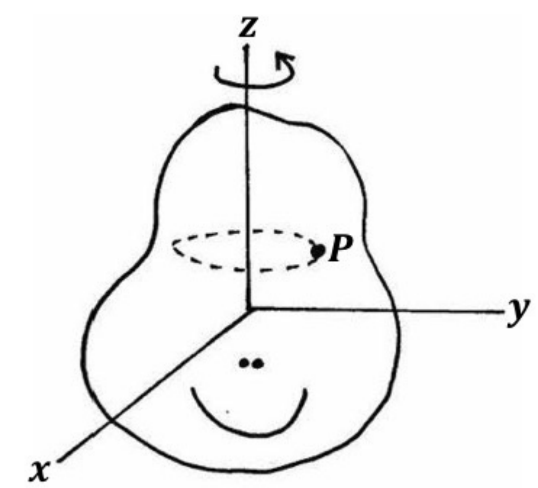
The balancing bird toy has its center of gravity located at the tip of its beak, this is why it can maintain equilibrium when balanced with a finger

Rotation and Rolling

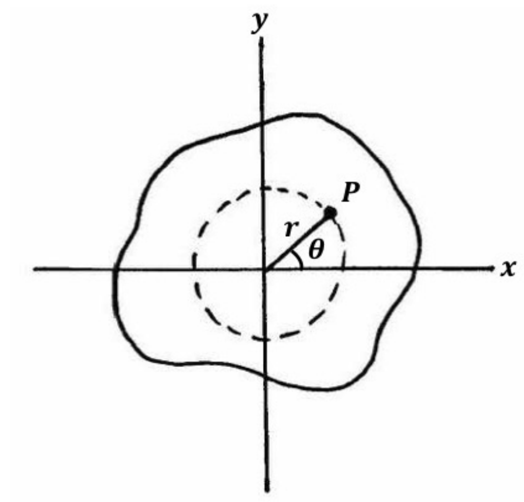


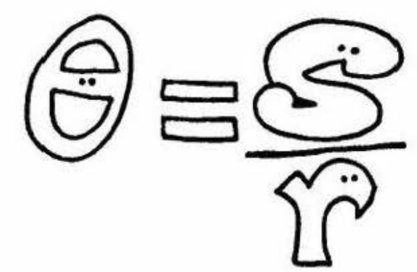
Rotation of a Rigid Object about a Fixed Axis:

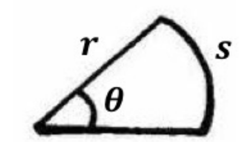
When an object rotates about a fixed axis that is perpendicular to a fixed plane, the motion is referred to as pure rotational motion. Every particle in the object will rotate in its own circle, but the center of all these circles coincide at the axis of rotation. A slice of the body in the x-y plane is shown.



The angular displacement θ of a point P is the arc length s divided by the radius of the circle r







Its unit is radian (rad). One radian is the angle subtended by an arc of length that is equal to the radius of the circle. Also $1\ \mathrm{rad} = 57.3^{\circ}$ (degrees) and $1\ \mathrm{rev} =$

$360^\circ = 2\pi\,\mathrm{rad}$. We may also write the angular displacement as $\Delta\theta = \theta_2 - \theta_1$.

The average angular speed is the angular displacement divided by the time in which that displacement took place

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

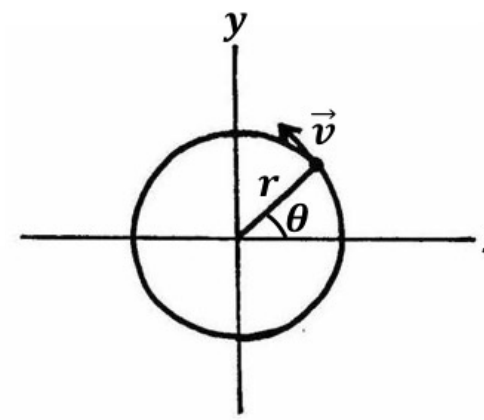
Its unit is rad/s. The average angular acceleration is the change in the angular speed divided by the time in which that change took place

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

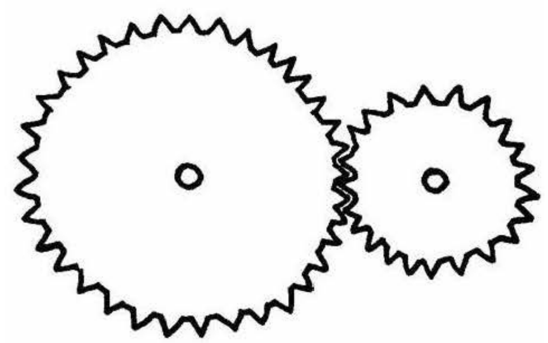
For a body in pure rotational motion, all particles will rotate through the same angle in a given time interval. Therefore, all particles have the same angular speed and the same angular acceleration, i.e. ω and α describe the motion of the whole object.

The linear velocity of a point in the body is found from $ds/dt = r\,d\theta/dt$ which gives

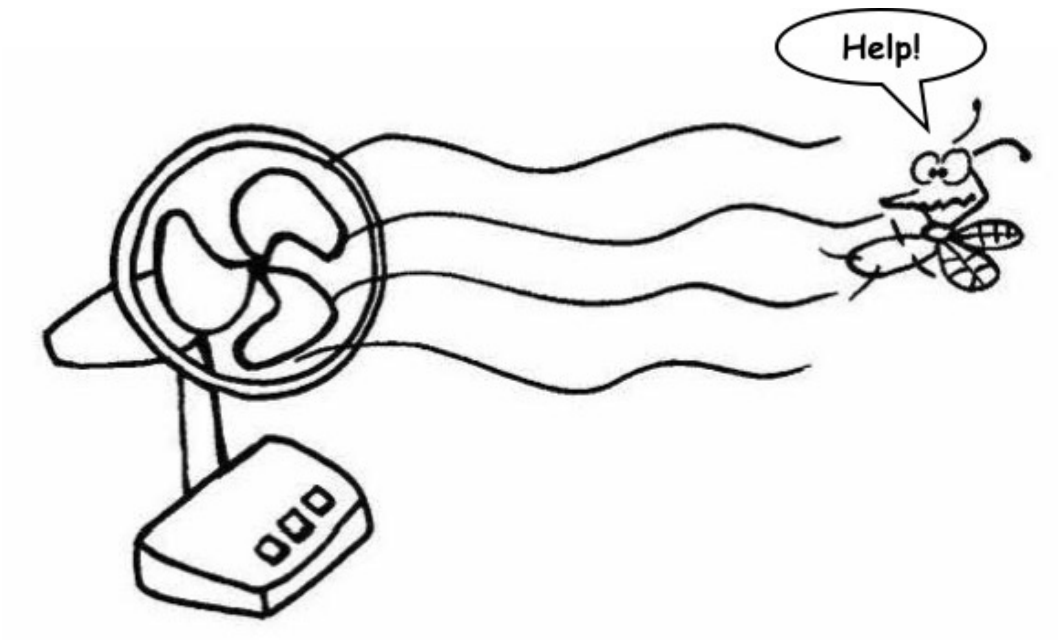
$$v = r\omega$$



This means particles at the same distance r from the rotation axis all have the same linear speed and the farther the particle is the greater its linear speed. This is how two sprockets can rotate together.



A typical fan rotates at an angular speed of 1500 rev/minute and has a diameter of about 0.6 m, and therefore the linear speed of the tip is about 50 m/s. The fly is experiencing it firsthand!



The rotational KE of the body is

$$K_R = \frac{1}{2} I \omega^2$$

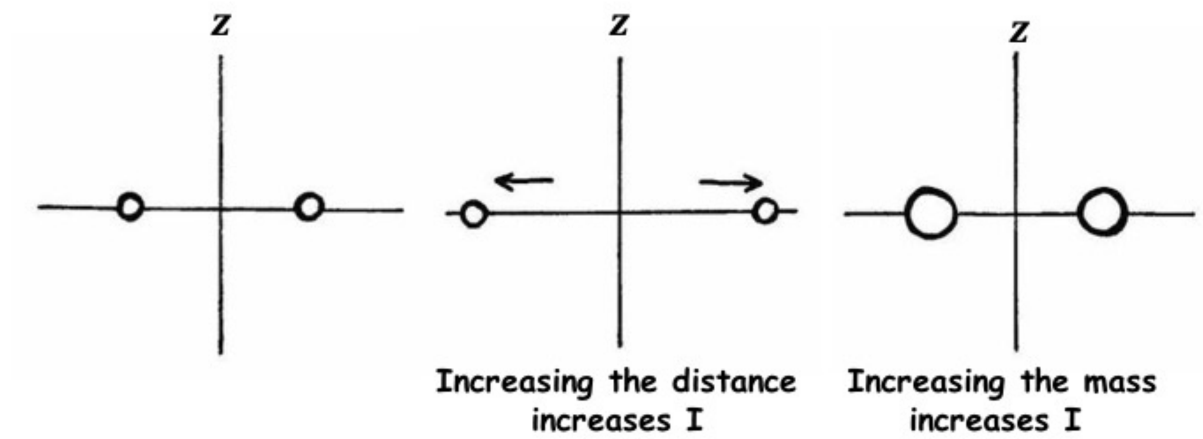
where I is known as the moment of inertia and it measures how the mass of the system is distributed about the axis of rotation. For a discrete system of particles, the moment of inertia is

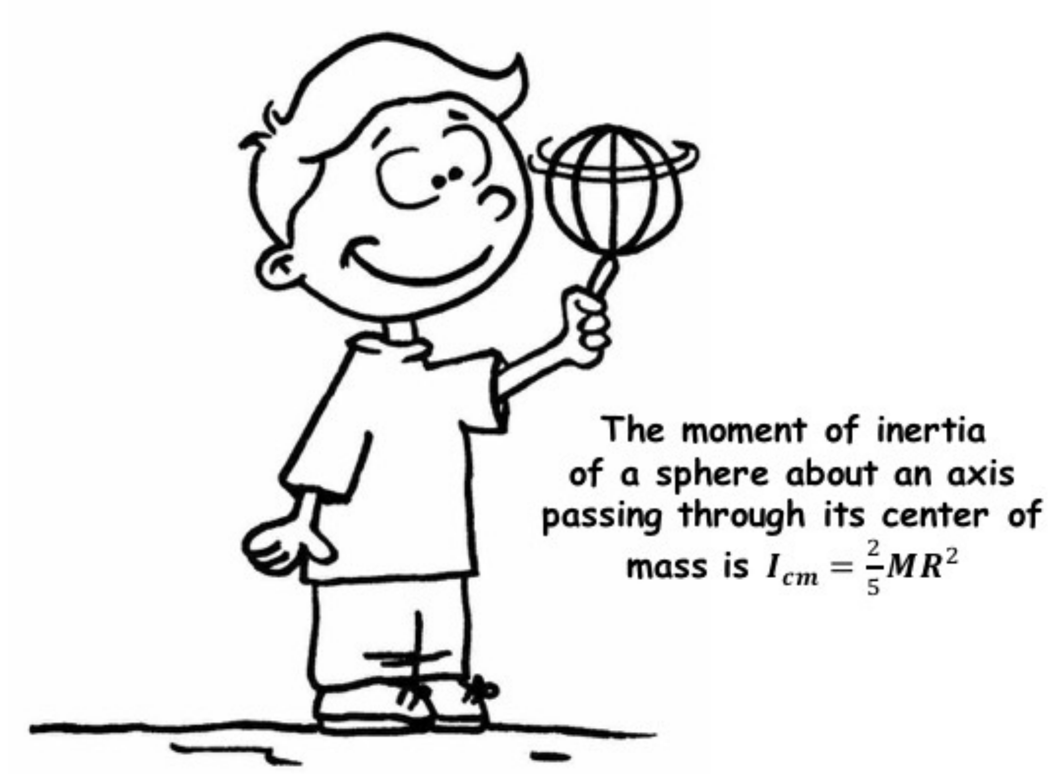
$$I = \sum_i m_i r_i^2$$

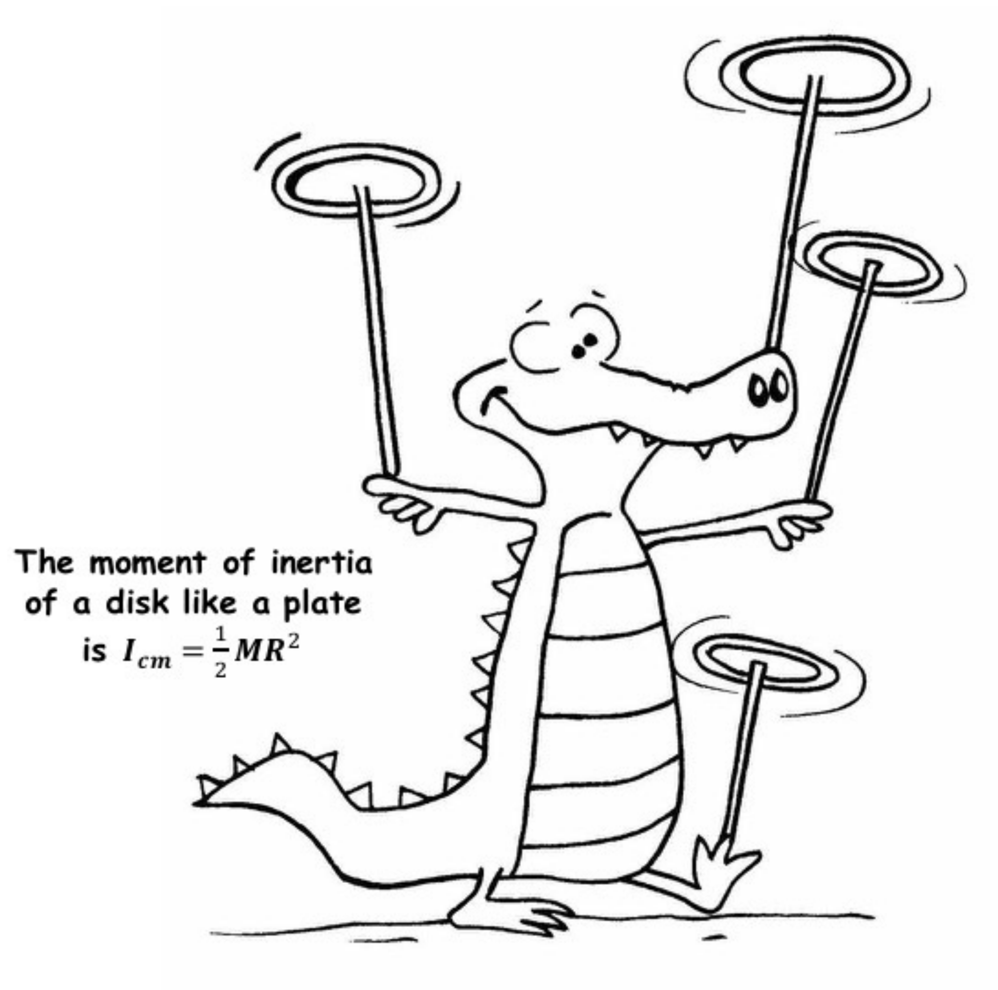
For a rigid body

$$I = \int r^2 \, dm$$

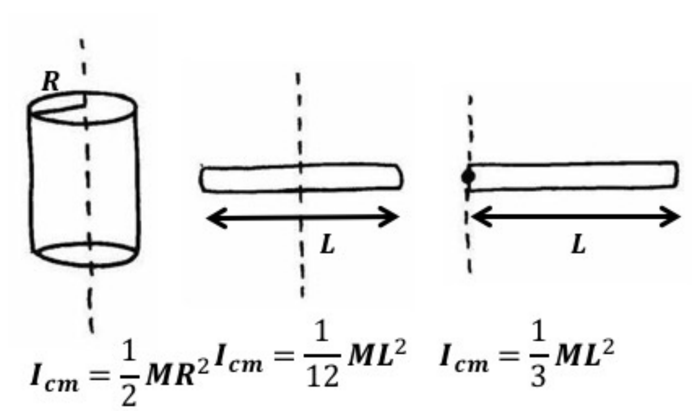
For a two-particle system rotating about the z-axis, to increase the moment of inertia, we may either increase the distance between the particles or increase the mass of one or two particles, or do both.







The moment of inertia of other objects calculated about the dotted axis is shown below



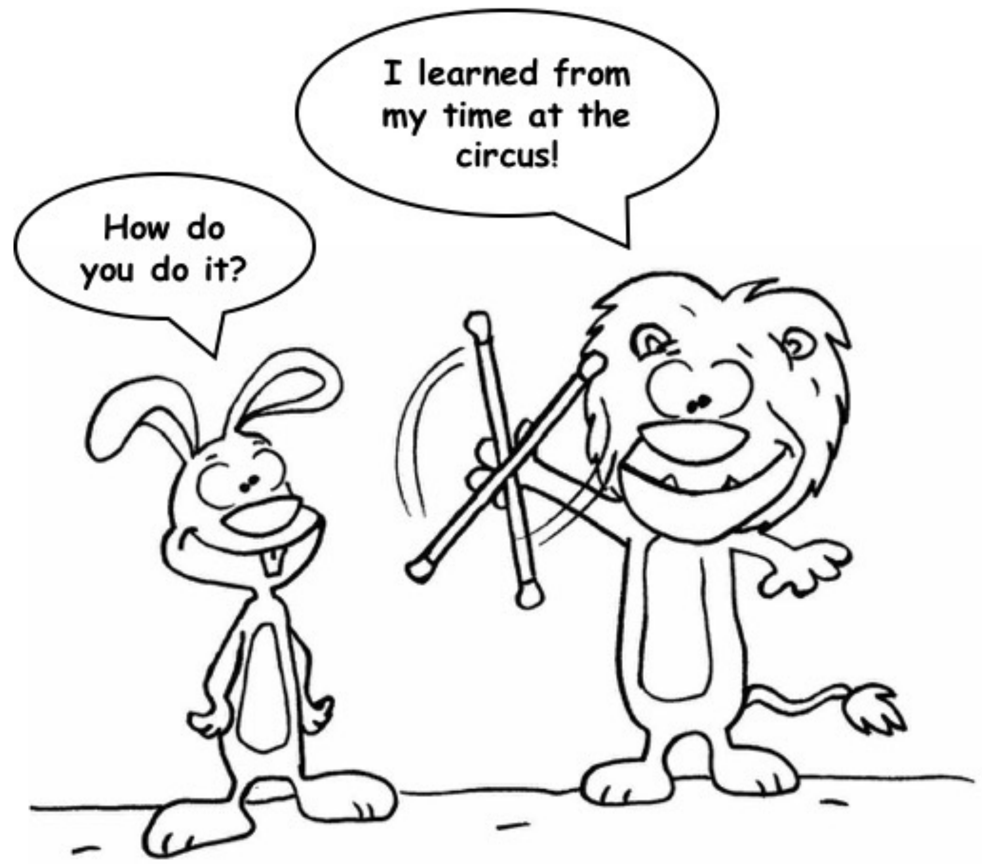
The total angular momentum of a symmetrical homogenous body in pure rotation about its symmetrical axis is

$$\vec{L} = I\vec{\omega}$$

and so, using $\sum \vec{\tau} = \frac{d\vec{L}}{dt}$, the torque is given by

$$\sum \vec{\tau}_{ext} = I\vec{\alpha}$$

Therefore, I represents the rotational analogue of mass, i.e. it measures how difficult it is to rotate the object.



Suppose that Bud pulls a rope wrapped around a cylindrical shell that is free to rotate about its central axis. The force generates a torque about that axis given by

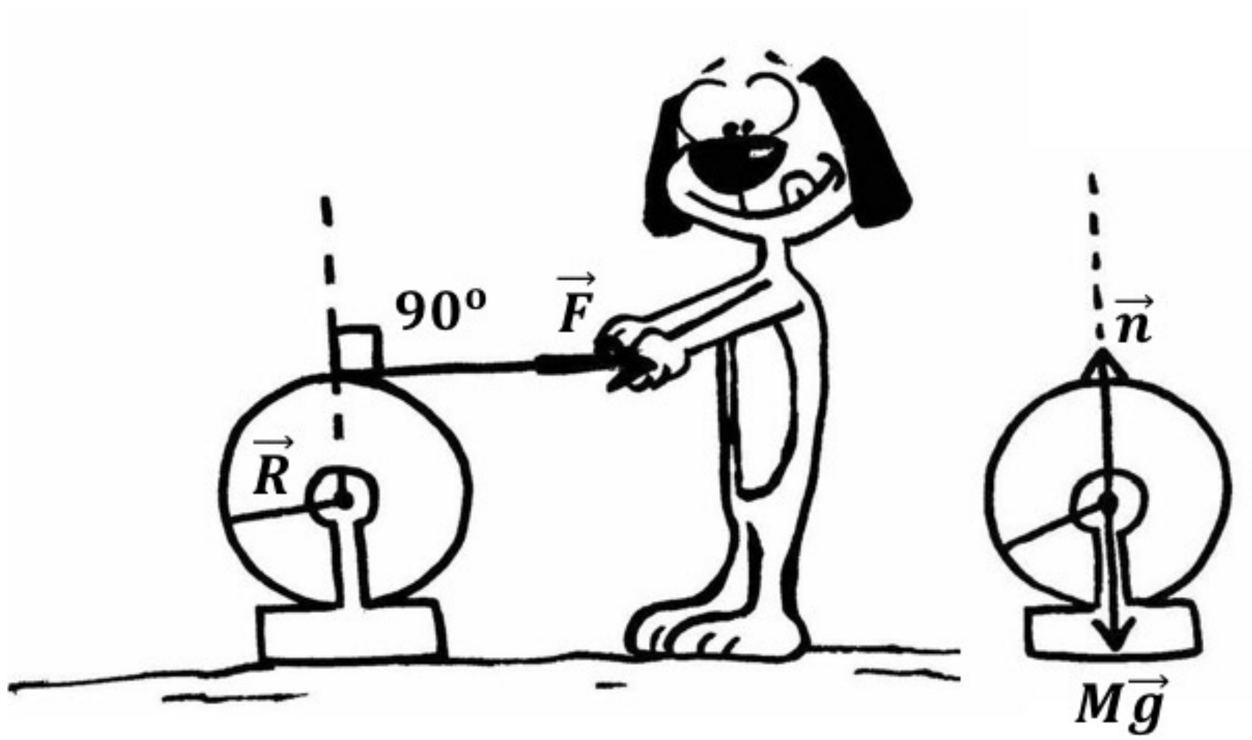
$$\tau = RF \sin 90^o = RF$$

The moment of inertia of a cylindrical shell is

$$I = MR^2$$

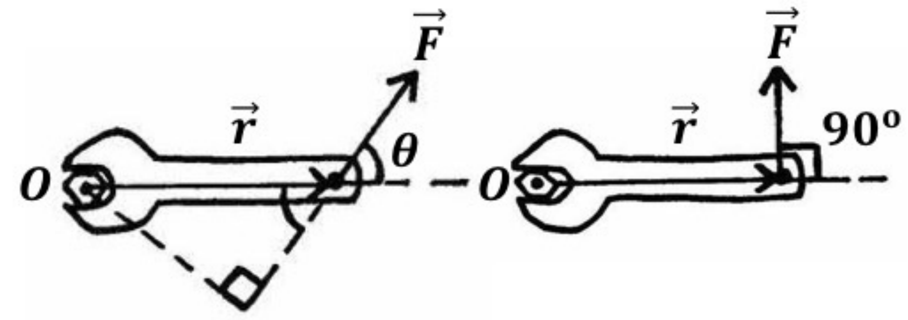
from this you may find the angular acceleration of the cylinder from

$$\alpha = \frac{\tau}{I}$$

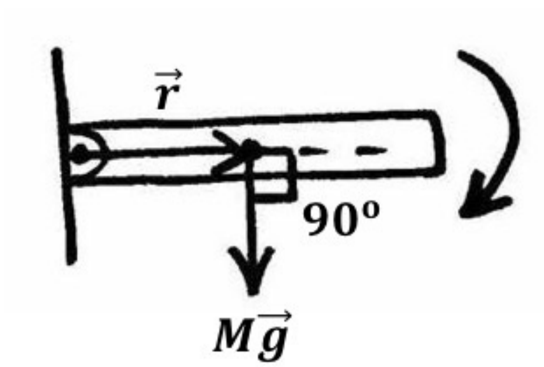


The torque done by either of the weight or the normal force about the axis of rotation passing through the center of the cylinder is zero, since their line of action passes through that axis (the angle between $\vec{r}$ and $\vec{F}$ is zero and so $\vec{\tau}$ is equal to zero).

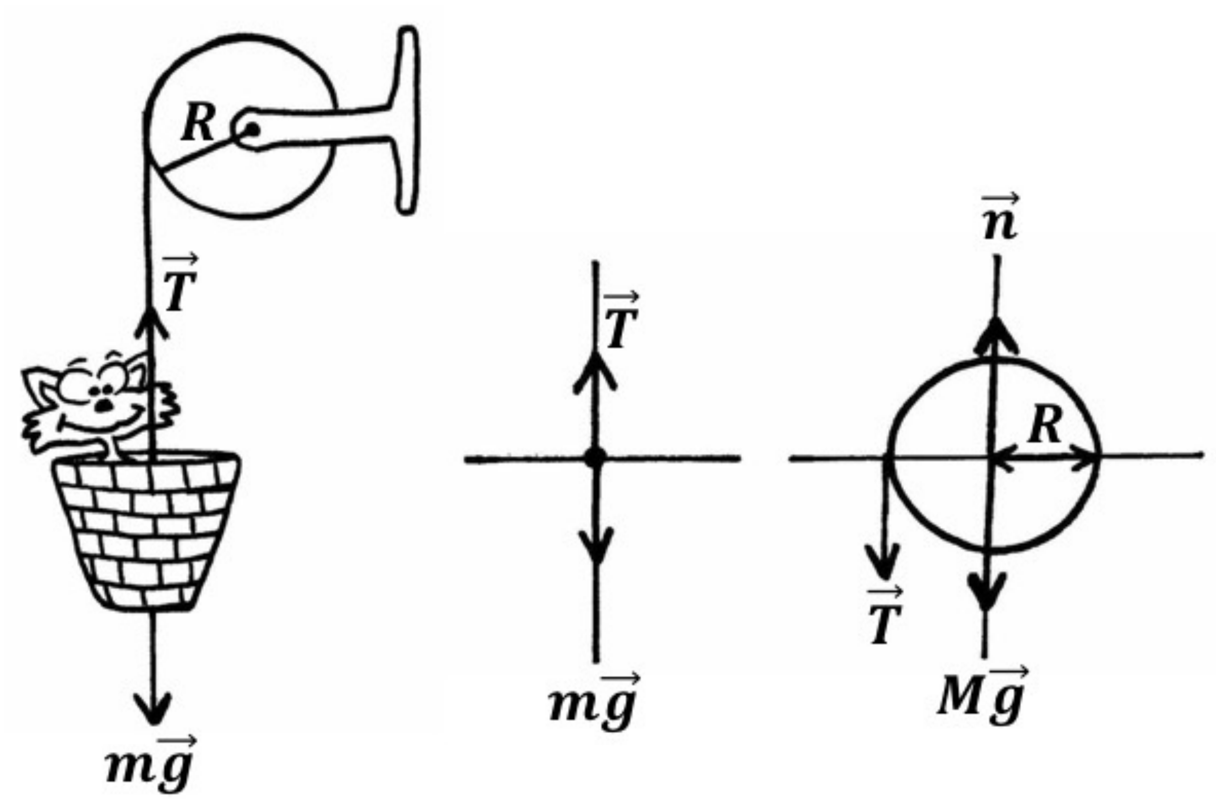
Another example of torque is the force applied to a wrench to loosen a bolt which produces a torque about O, this torque is maximum if the angle between $\vec{r}$ and $\vec{F}$ is 90°.



An additional example is the torque, $\tau = rF \sin 90^o = rMg$, produced by the weight of a rod that is free to rotate about its end point as shown. Using $I = \frac{1}{3}ML^2$, you may find the angular acceleration of the rod from $\alpha = \tau/I$.



Suppose a cat is in a basket descending that is attached to a rope wrapped around a cylinder free to rotate about its center axis, the tension in the rope will then produce a torque about that axis. Thus, as the cat accelerates downwards, the cylinder rotates with an angular acceleration of $\alpha = \frac{TR}{I}$. Note that the acceleration a of the (basket + cat) is equal to the (tangential) acceleration of a point at the rim of the cylinder, i.e. $a = R\alpha$. The free body diagrams of each of the cat and the cylinder are shown. By applying Newton's second law, you may show that $\alpha = g/R(1 + M/2m)$, where m and M are the masses of the combined (basket + cat) and the cylinder respectively.



Conservation of the angular momentum of a Rigid Body in Pure Rotational Motion:

If the component of the external torque about the rotation axis is zero (say the z-axis), then the component of angular momentum along that axis remains constant (conserved).

If

$$\tau_z = \frac{dL_z}{dt} = 0$$

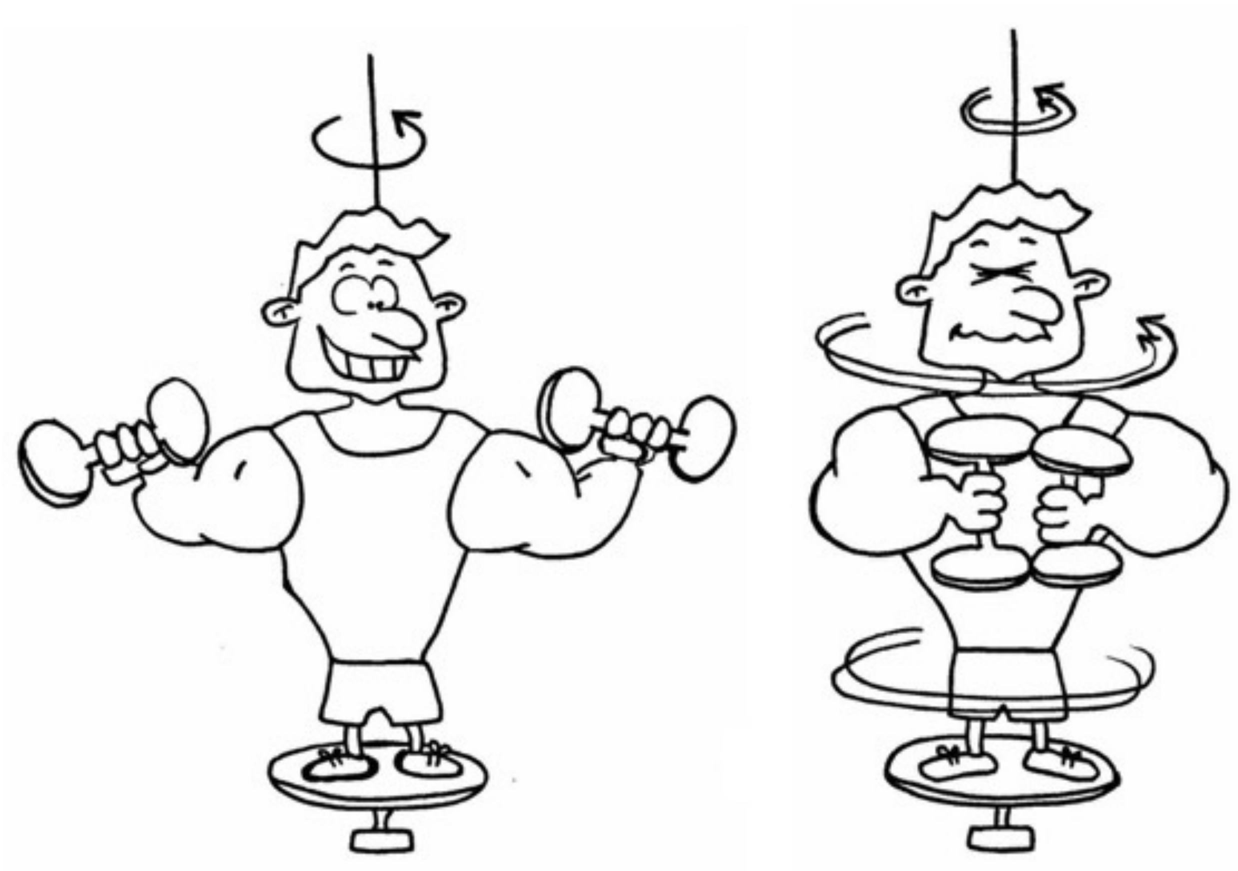
then

$$L_{zi} = L_{zf}$$

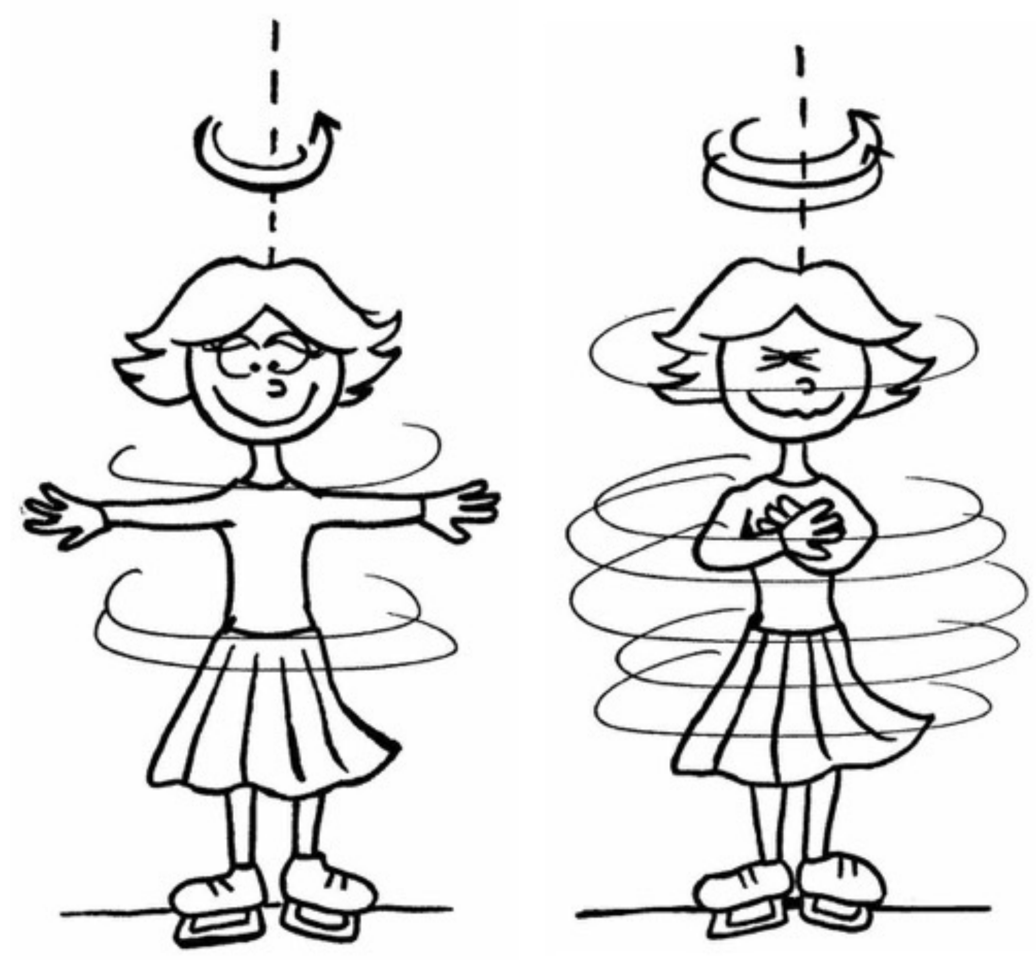
or

$$I_i\omega_i = I_f\omega_f$$

Suppose a bodybuilder is standing on a platform that is free to rotate about a vertical axis passing through its center. If he is initially rotating with an angular speed of ω_i, and if we neglect friction, we may assume that the net external torque acting on the system about the vertical axis is zero. Therefore, the angular momentum is conserved along that axis, which means the angular speed will increase if the man draws the weighs in (decrease the moment of inertia) according to $\omega_f = I_i\omega_i/I_f$ and he will spin faster. This increase in the KE is because the bodybuilder does work when he draws the weighs in.

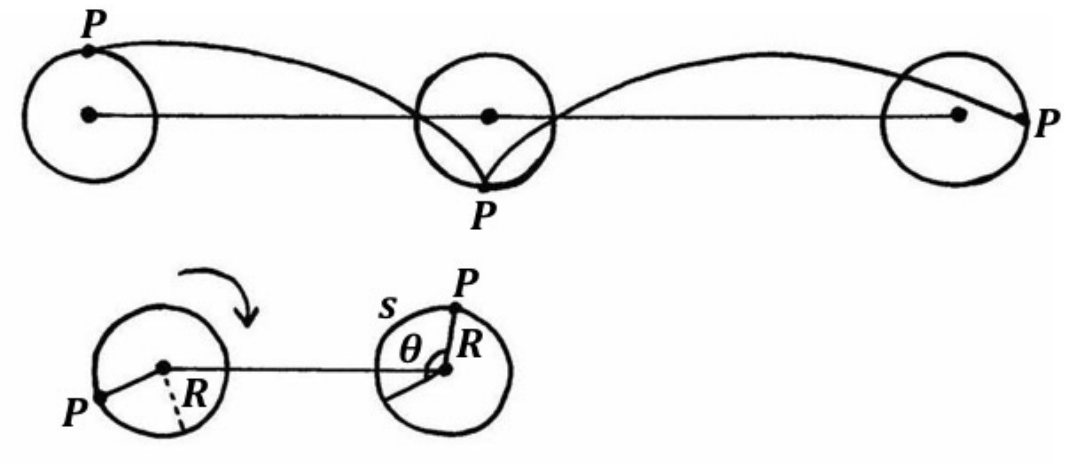


The same situation happens to a figure skater who is initially rotating with an angular speed of ω_i. By ignoring friction between the skater and the ice, the net external torque on the skater is zero and therefore as she moves her hands close to her body, her moment of inertia decreases which will increase her angular momentum.

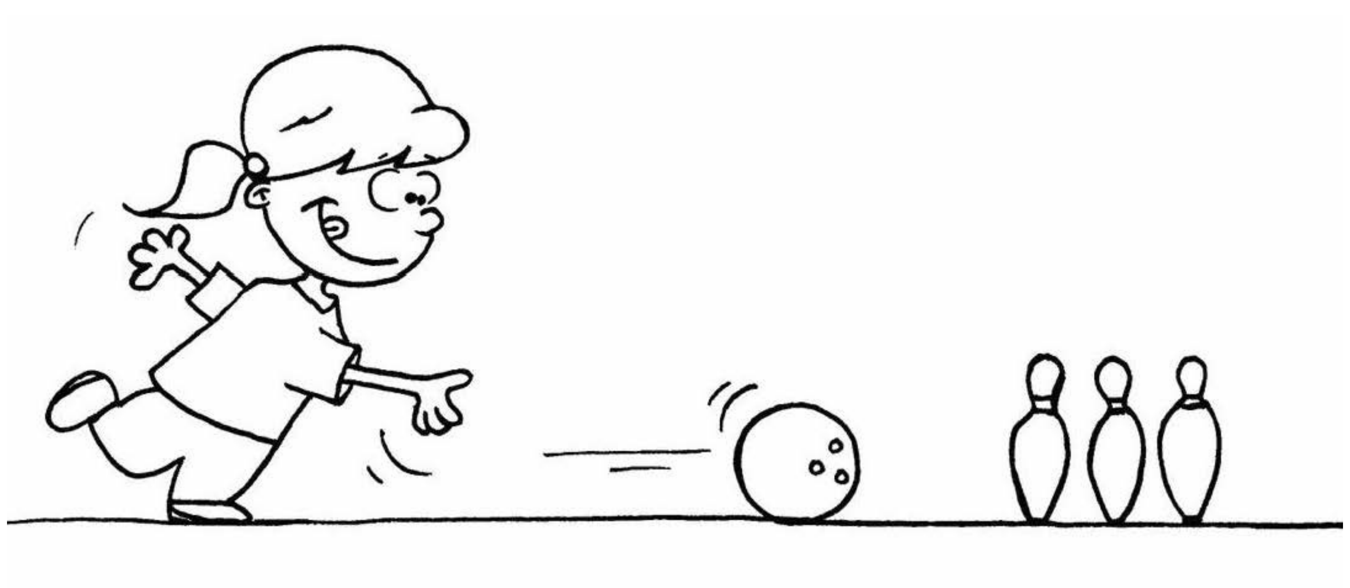


Rolling Motion:

Rolling motion can be considered as a combination of pure translational motion of the center of mass and a pure rotational motion about an axis passing through the center of mass perpendicular to the plane of motion. For example, the center of mass of a cylinder moves in a straight line, while a point at the rim moves in a cycloid path as shown.



Some examples are the motion of a balling ball or a bicycle wheel.

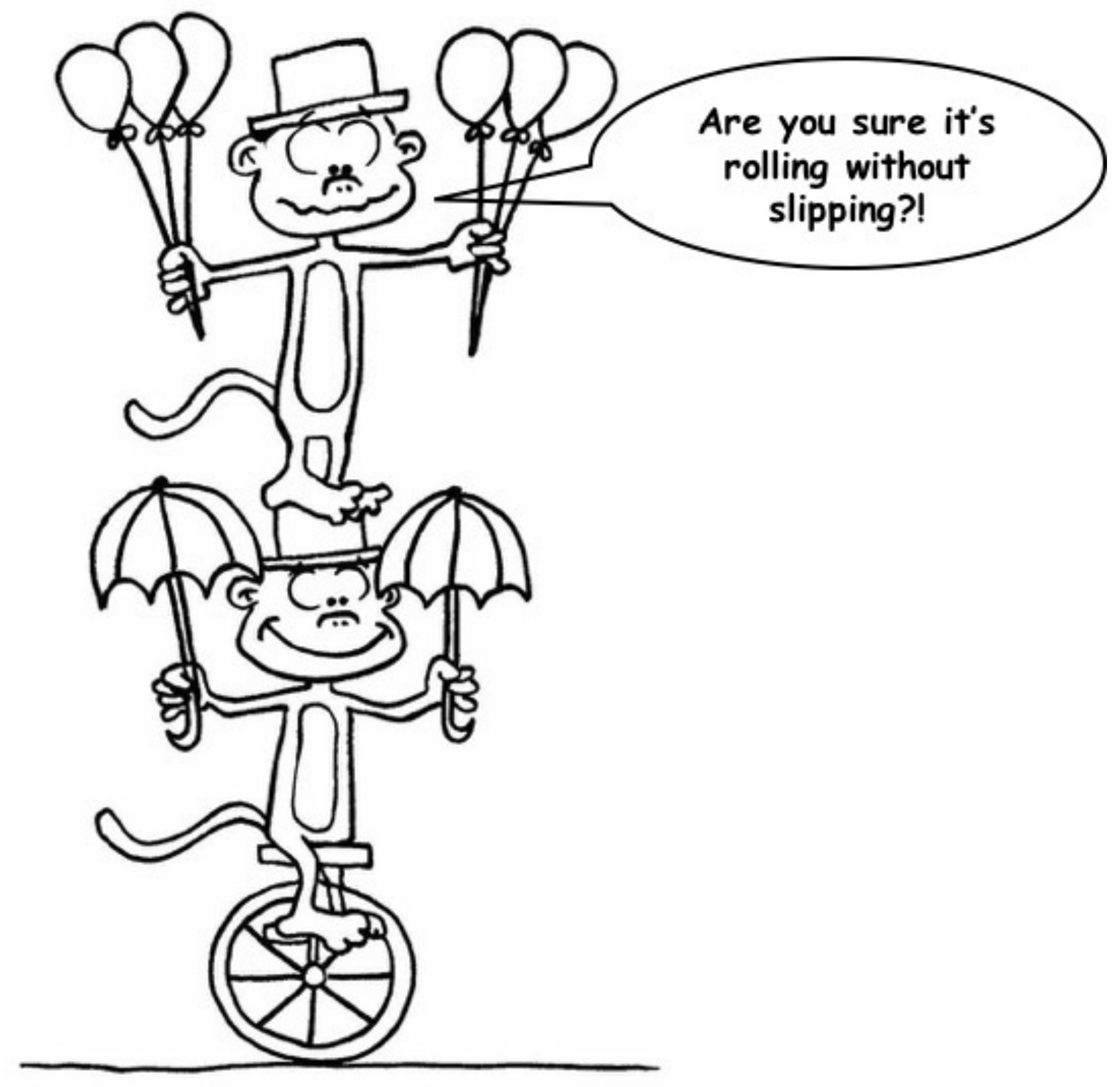


If the object rolls without slipping (such as when a perfectly rigid object rolls on a perfectly rigid surface), then the instantaneous point of contact between the object and the surface is always at rest relative to the surface. This means that when the center of mass moves a distance s, a point at the rim would move the same distance of arc length s, and so we have

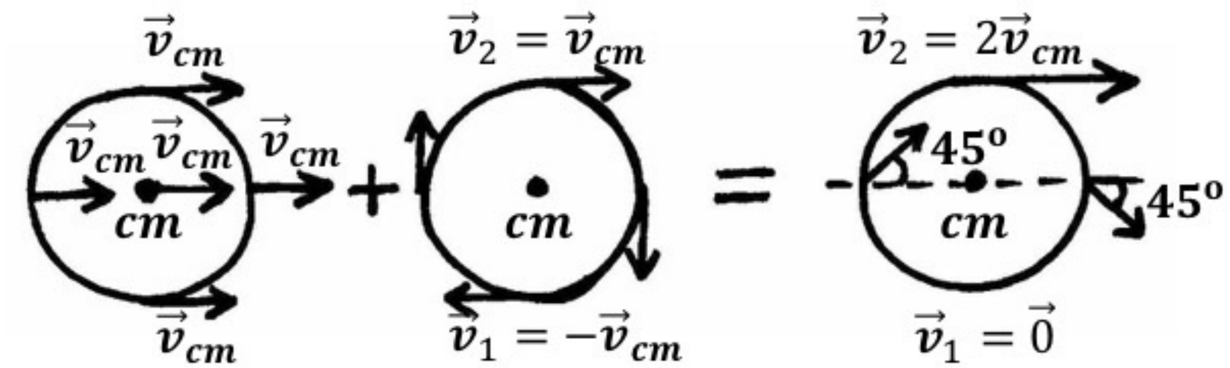
$$v_{cm} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

and

$$a_{cm} = \frac{dv_{cm}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



The figure below shows how the linear velocity of each point is equal to its velocity due to the translational motion and its velocity due to the pure rotational motion. You can see how the instantaneous point of contact has zero velocity, while that at the top has a velocity of $\vec{v}_2 = 2\vec{v}_{cm}$.

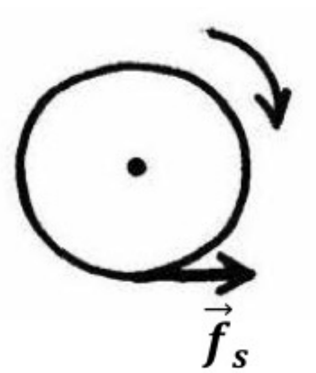


The total KE of the rolling object is

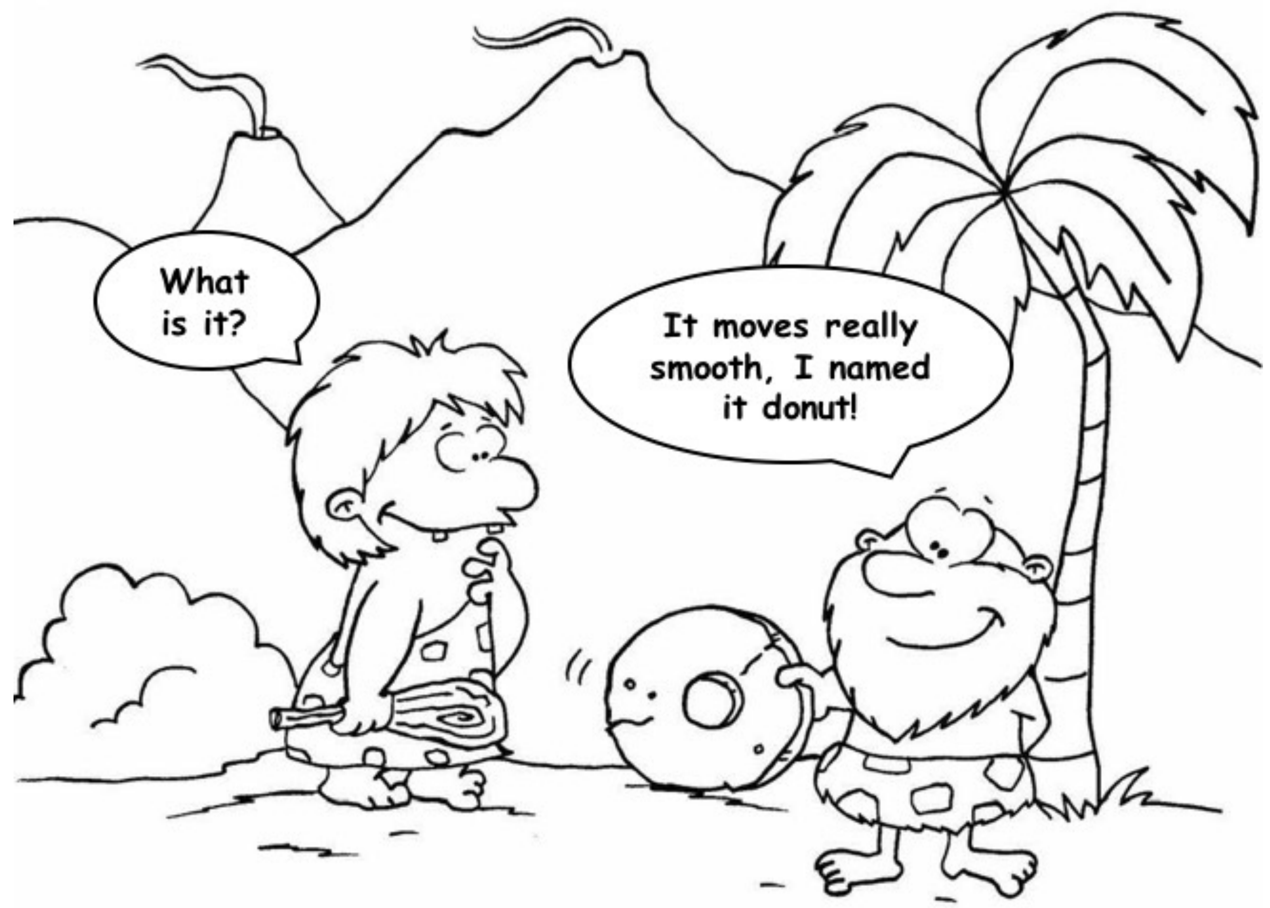
$$K_R = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_{cm}^2$$

$$= \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

The force that provides the necessary torque for the object to roll is the statistical frictional force f_s. Since the point of contact with the surface does not slide, the frictional force does no work (as there is no displacement of that point) and so it can be assumed (if both surfaces are



rigid enough) that the total mechanical energy is not lost while the object rolls. This is why a wheel is a very efficient way of moving and is one of the most important inventions!



In a realistic situation, the body and the surface are not perfectly rigid and as a result the normal force is not a single force but a number of forces over the area of contact where their lines of action do not pass through the center of

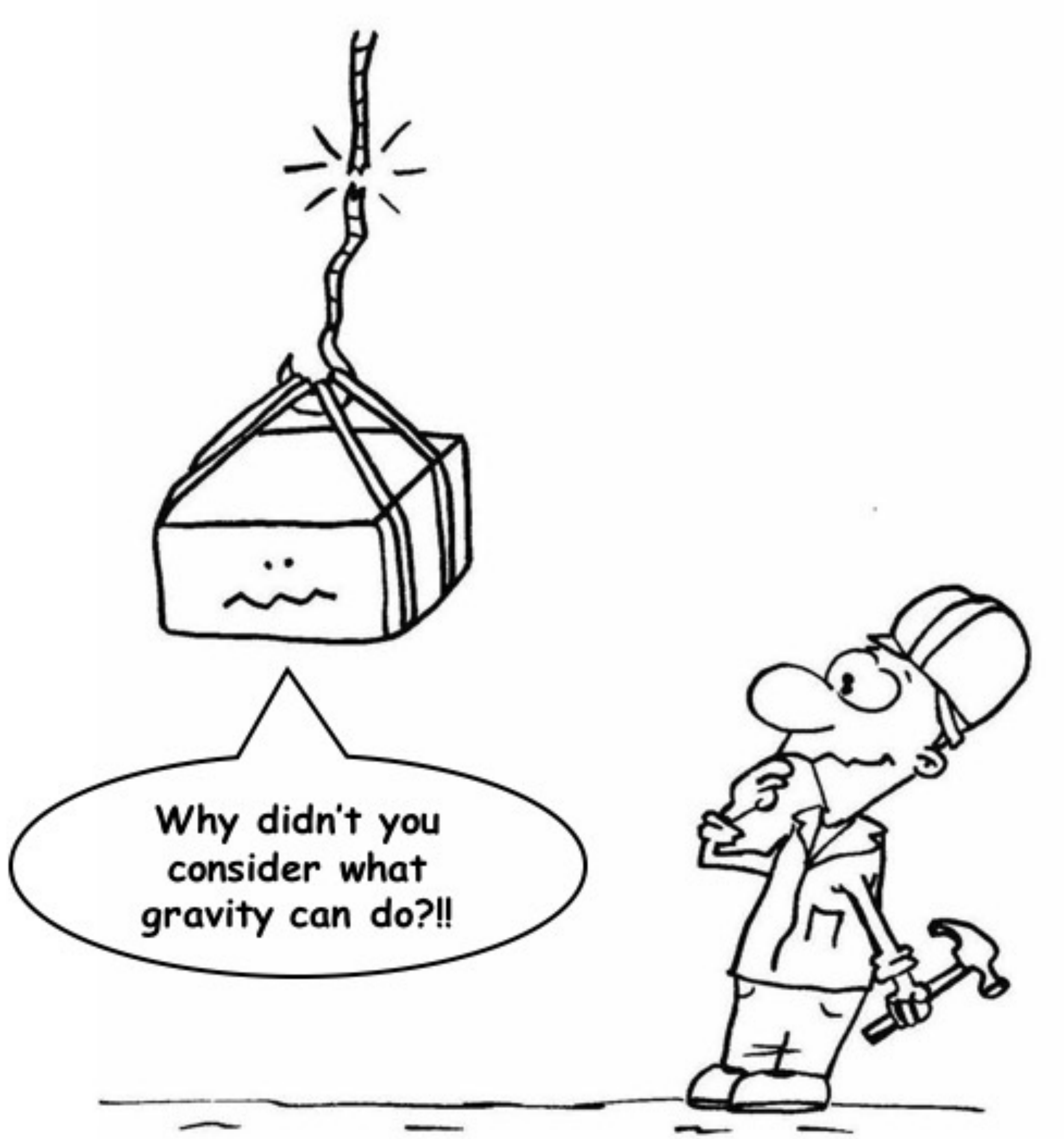
mass and therefor will produce a torque that opposes the rotation. In addition, due to the presence of deformation, there can be some sliding of the object over the surface which will result in a loss of mechanical energy. These two effects combined is known as rolling friction.

Newton's Law of Gravity

Newton's Law of Gravity:

In 1687, Sir Isaac Newton made the remarkable discovery that the force holding the moon or a planet around the sun is the same force that makes an apple fall from a tree.

This law states that: ***every particle in the universe attracts every other particle with a force that is directly proportional to the product of the masses and inversely proportional to the square of the distance between them.***



If two particles have masses of m_1 and m_2, and the distance between them is r, then the magnitude of the gravitational force is

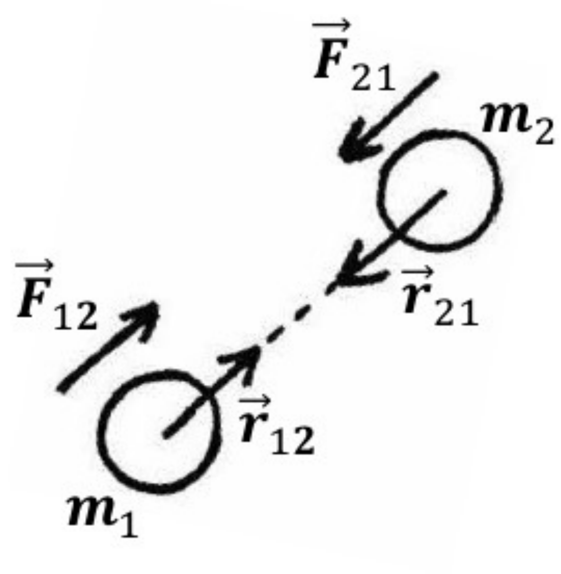
$$F_g = \frac{Gm_1m_2}{r^2}$$

where G is the universal gravitational constant that has the value

$$G = 6.674 \times 10^{-11}\ \mathrm{N.m^2/kg^2}$$

The direction of this force is along the line joining the two particles as shown

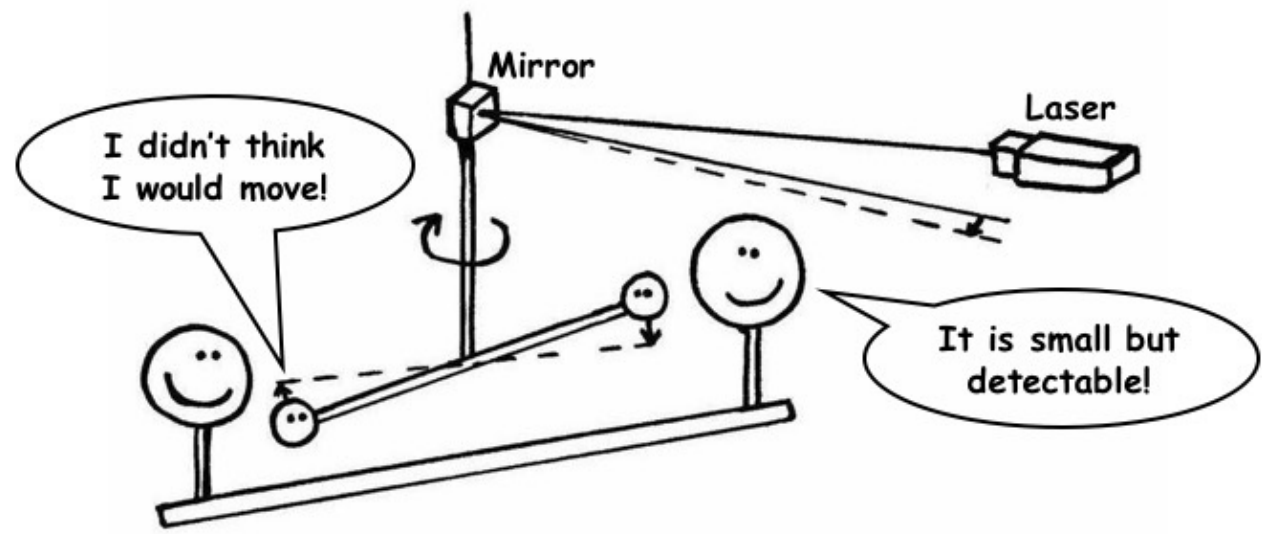
$$\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\vec{r}_{12}$$



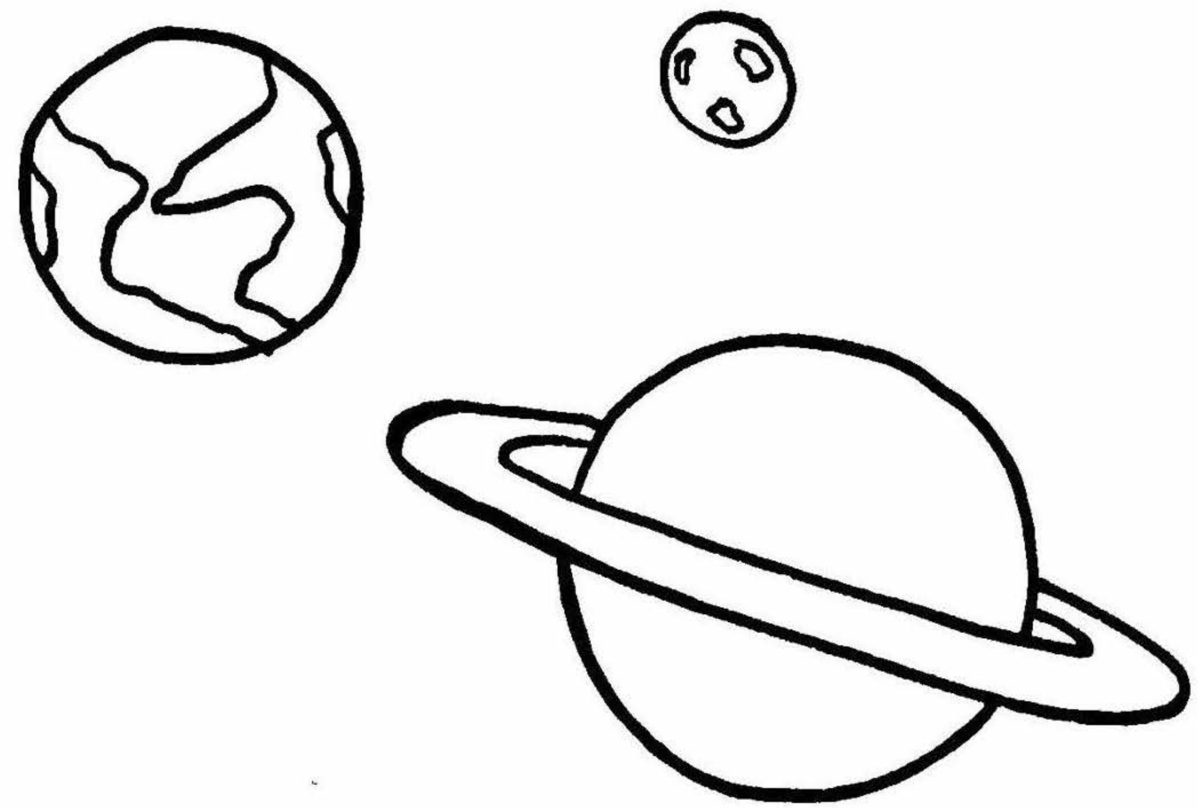
where $\vec{r}_{12}$ is a unit vector directed from m_1 to m_2. The negative sign indicates that the force is attractive. Note that each particle exerts a gravitational force on the other and the two form an action reaction pair, i.e.

$$\vec{F}_{12} = -\vec{F}_{21}$$

The value of G was determined in the late 19th century using an apparatus known as the torsion balance which was constructed by Henry Cavendish in 1798 to measure the density of Earth. This apparatus consists of two small spheres, each of mass m, placed at the ends of a light horizontal rode supported by a thin vertical quartz fiber forming an inverted T (this is the modern version of the Cavendish apparatus). Two larger spheres, each of mass M, are placed at the positions shown. The attractive gravitational force between the smaller and larger spheres causes the T to rotate (in this case clockwise) through a small angle. This angle is measured by measuring the deflection of a light beam reflected from a mirror fixed at the quartz fiber (the deflection angle is enlarged in this figure for clarity).



The force of gravity is what gives planets their spherical shape, this is because as the mass of an object becomes very large, all particles from all sides are attracted evenly towards the center.



It can be shown that when a spherically symmetric mass distribution (such as a planet or other celestial objects) interact with a particle outside of it, the spherical distribution behaves as if its entire mass is concentrated at its center. Therefore, the gravitational force between such sphere and a particle is

$$F_g = \frac{GMm}{r^2}$$

where r here is the distance from the particle to the center of the sphere. For example, if the planet is Earth, then this force is

$$F_g = \frac{GM_E m}{r^2}$$

where M_E is the mass of the Earth. From the definition of weight, we know that this force is just the weight of an object, and so

$$w = mg_E = F_g = \frac{GM_E m}{r^2}$$

which gives

$$g_E = \frac{GM_E}{r^2}$$

This shows that, as discussed previously, the free-falling acceleration g is independent of the mass of the object. If the object is falling near the Earth's surface, then we may write $r \approx R_E = 6.37 \times 10^6$ m and since $M_E = 5.98 \times 10^{24}$ kg, then the gravitational acceleration near the Earth's surface is

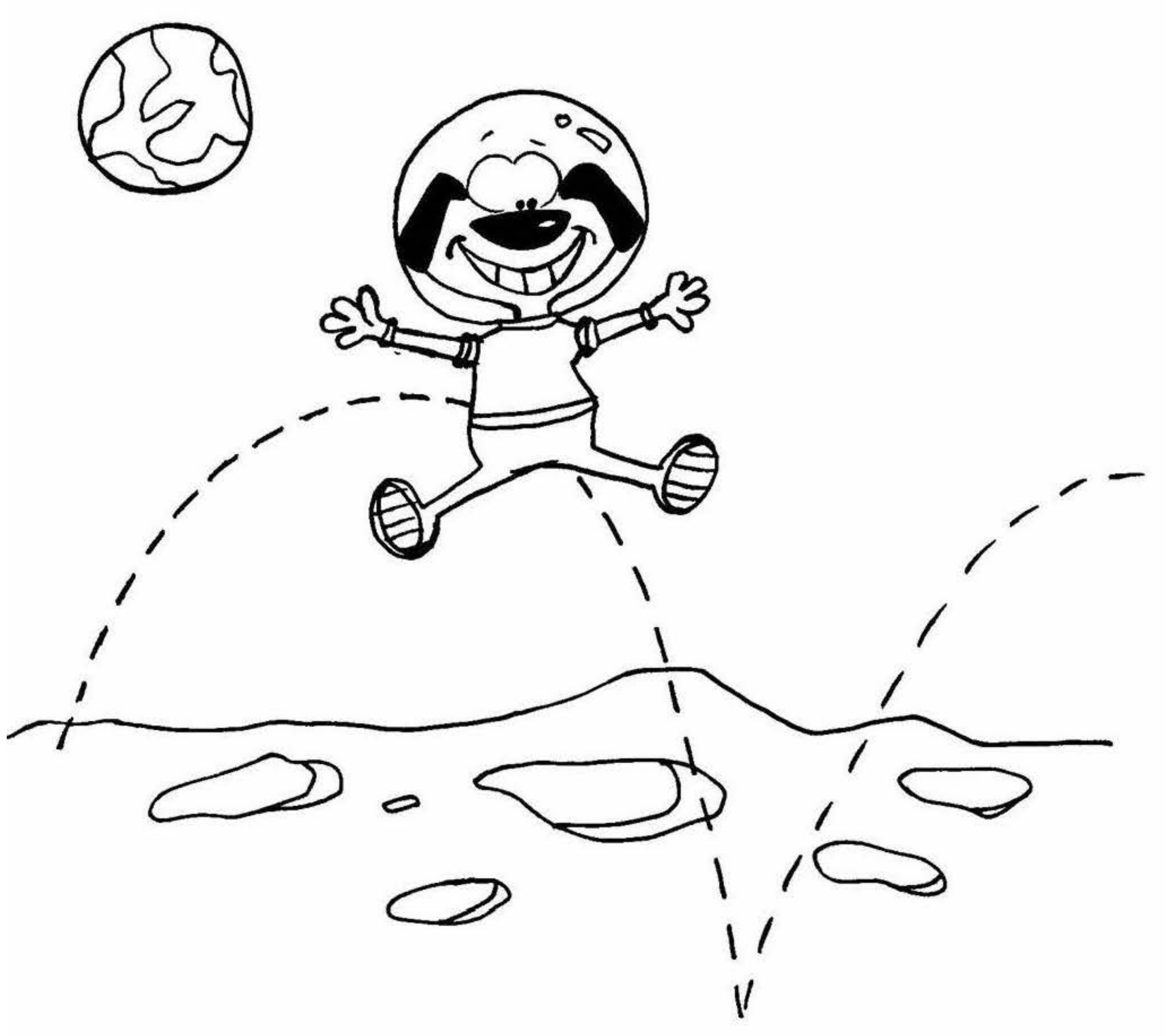
$$g_E \approx 9.8 \text{ m/s}^2$$

At an altitude h high above the ground we may write

$$g_E = \frac{GM_E}{(R_E + h)^2}$$

This shows why the weight of an object decreases with an increasing altitude.

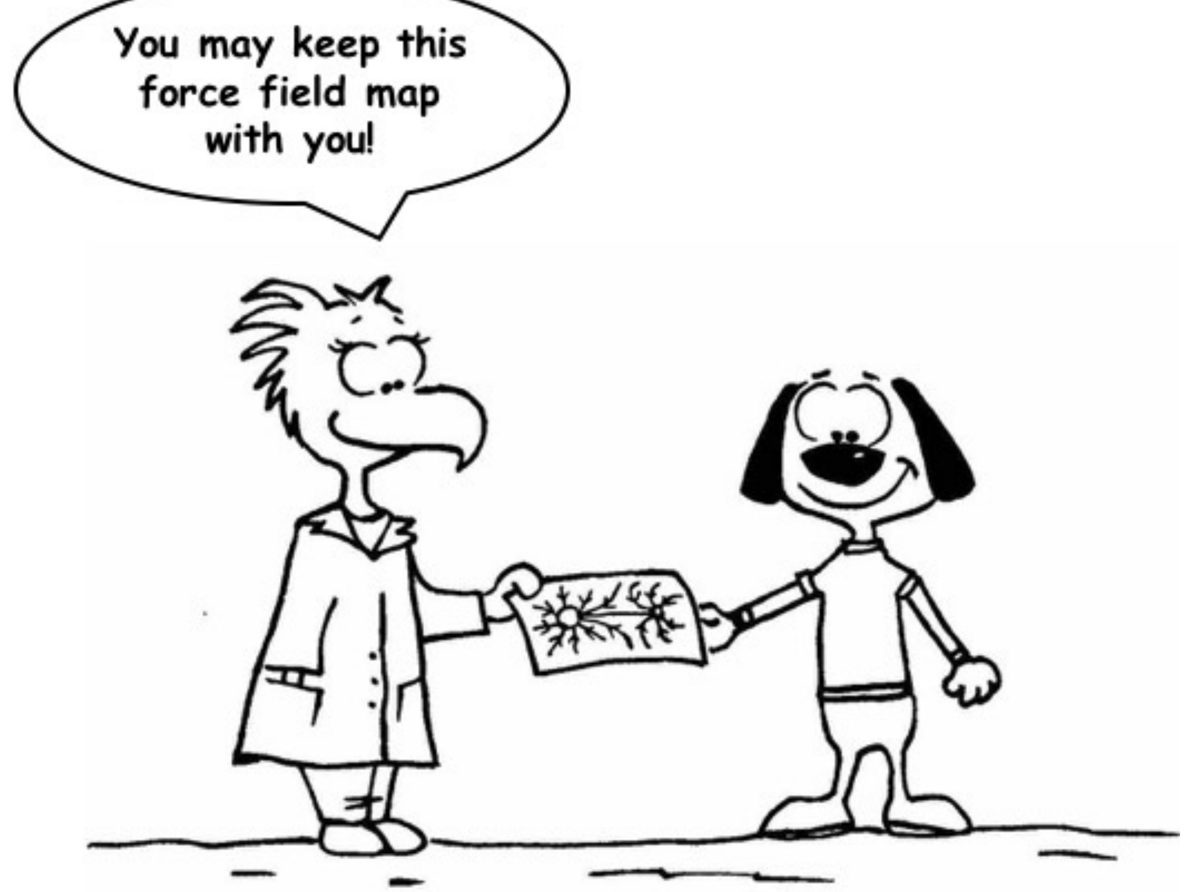
On the surface of the moon, the gravitational acceleration is much less than that on Earth and is only $g_m = 1.6\ \mathrm{m/s^2}$. So, if Bud jumps on the surface of the moon, then compared to a jump on Earth, given the same initial velocity, he will reach a maximum height of six times that on Earth.



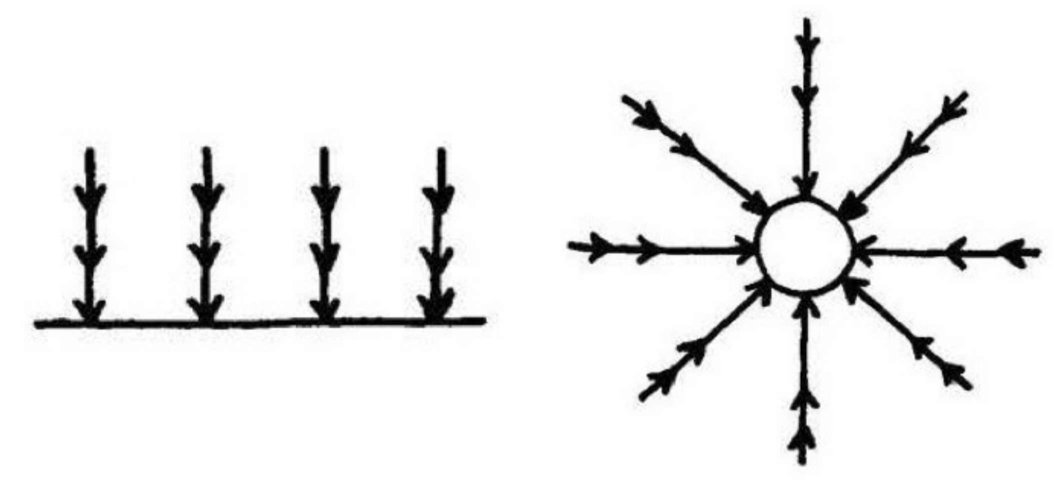
This brings us to the concept of the gravitational field. A small test particle of mass m_o will experience a gravitational force when placed at different points in the vicinity of another mass M. Therefore, we may consider M as producing a gravitational field around it that is given by

$$\vec{g} = \frac{\vec{F}_g}{m_o} = \frac{-GM}{r^2}\vec{r}_1$$

I.e. the gravitational field at a certain point is defined as the gravitational force per unit mass at that point. We can therefore draw a map of this field showing the gravitational field at any point around the mass M.

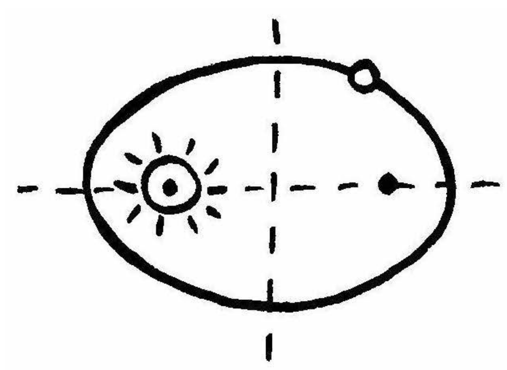


The gravitational field vectors near the Earth's surface and at large distances from the Earth are shown.



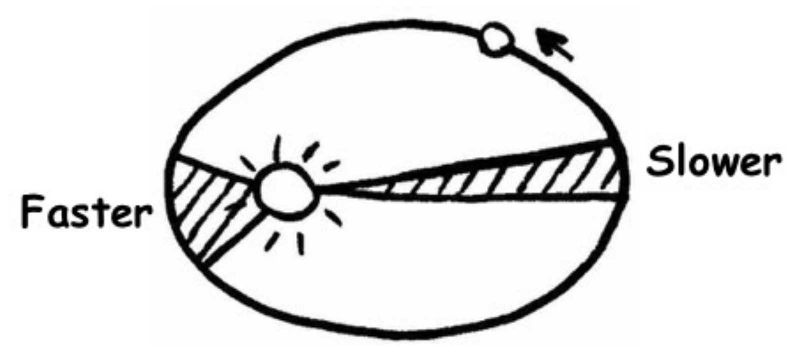
Kepler's First Law:

All planets move about the sun in elliptical orbits with the sun at one focus.



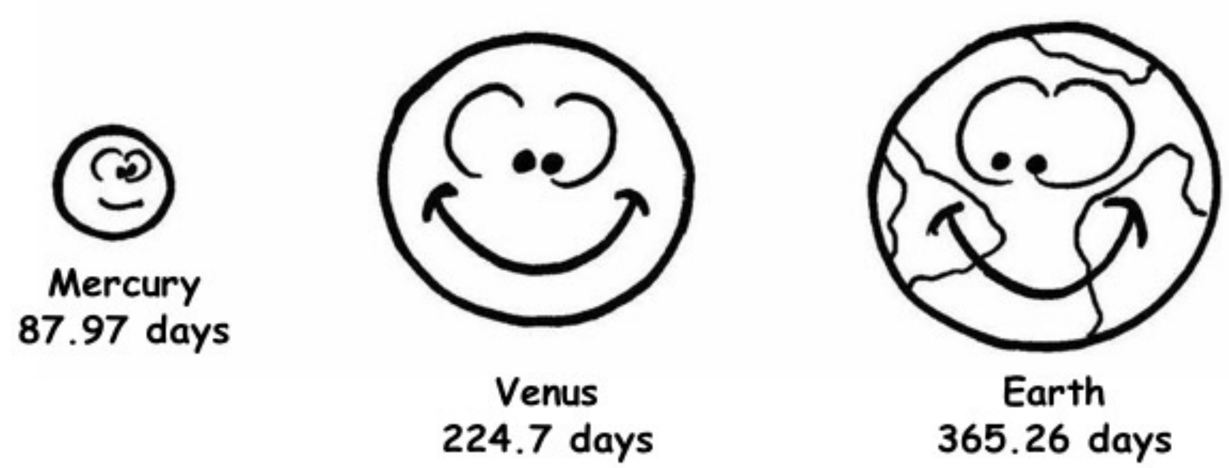
Kepler's Second Law:

The line from the sun to the planet sweeps out equal areas in equal times. Therefore, the planet speeds up as it moves towards the sun and slows down as it moves away from the sun.



Kepler's Third Law:

The square of the orbital period of the planet is proportional to the cube of the semi-major axis of its elliptical orbit

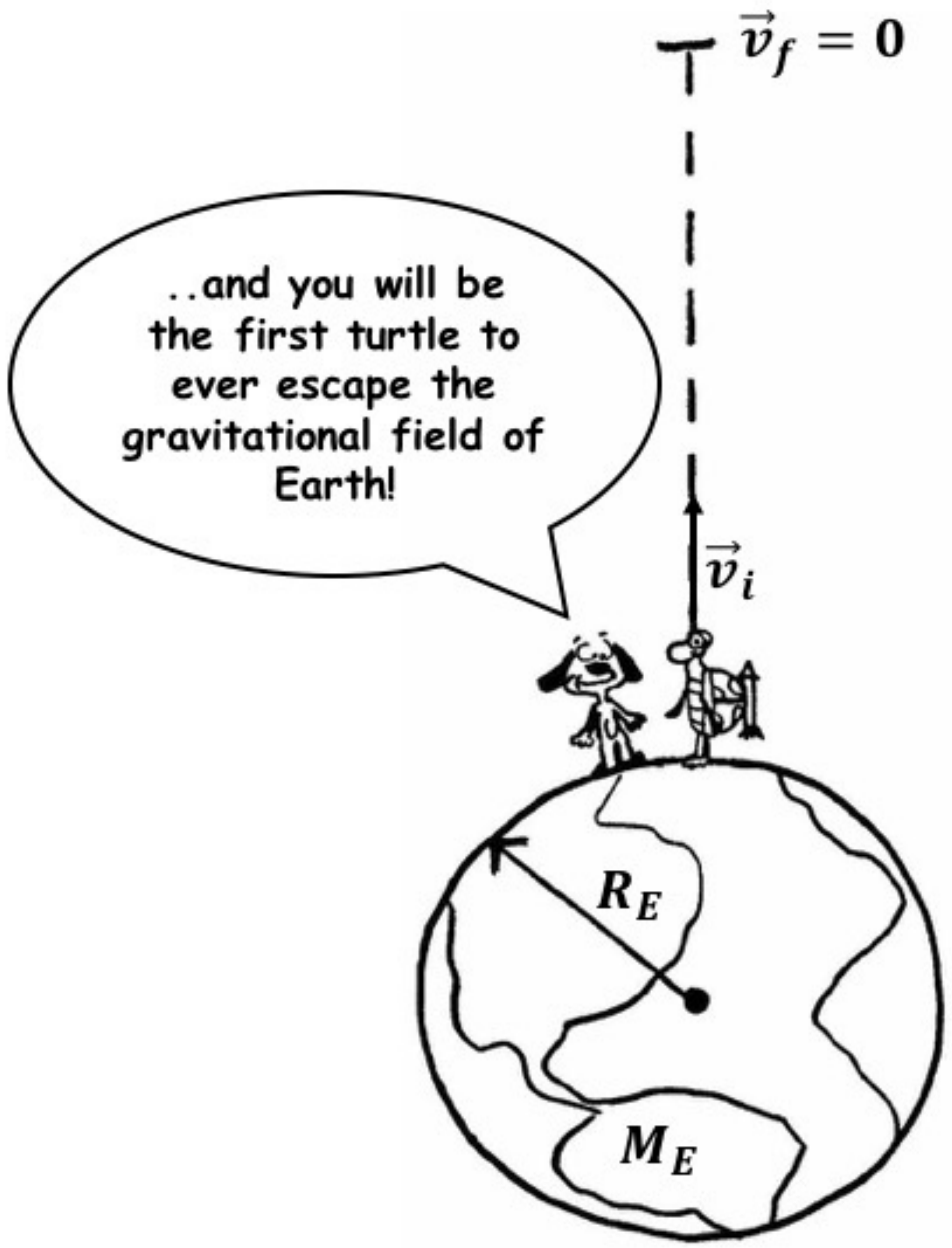


The Escape Speed:

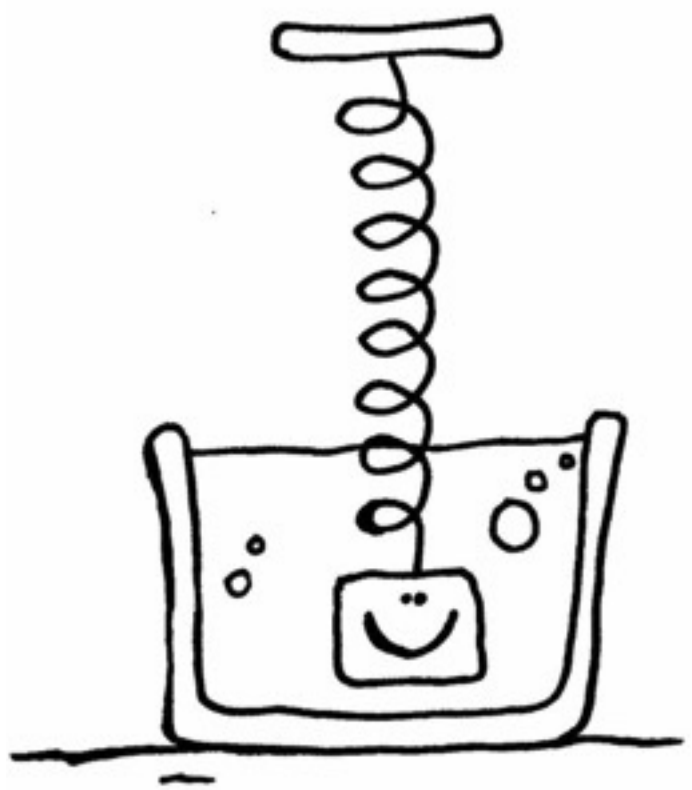
If an object of mass m is projected vertically upwards from a planet, then the minimum value of the initial speed of the object in order for it to continue to move infinitely far from the planet and escape its gravitational influence is

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

where M and R are the mass and radius of the planet respectively.

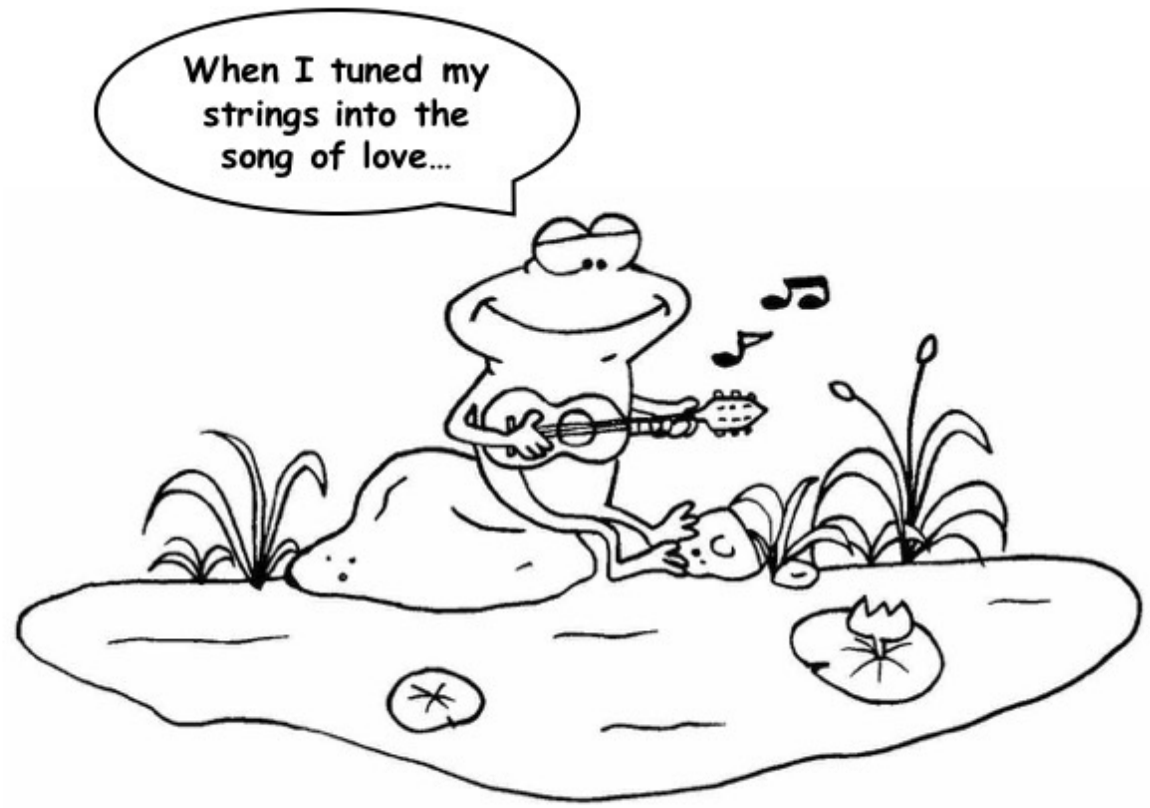


Oscillatory Motion



Simple Harmonic Motion:

A motion that repeats itself is known as periodic motion. Many physical systems exhibit such vibrational behavior. Some examples are the oscillations of an object attached to a spring, the swinging of a pendulum, a vibrating string in a musical instrument, the oscillations of the electric and magnetic field vectors in an electromagnetic wave and the periodic variations of voltage and current in an alternating current circuit. Therefore, vibrations are very important phenomena in physics and are found everywhere in nature. In fact, it is known that Nikola Tesla has said "If you want to find the secrets of the universe, think in terms of energy, frequency and vibration."



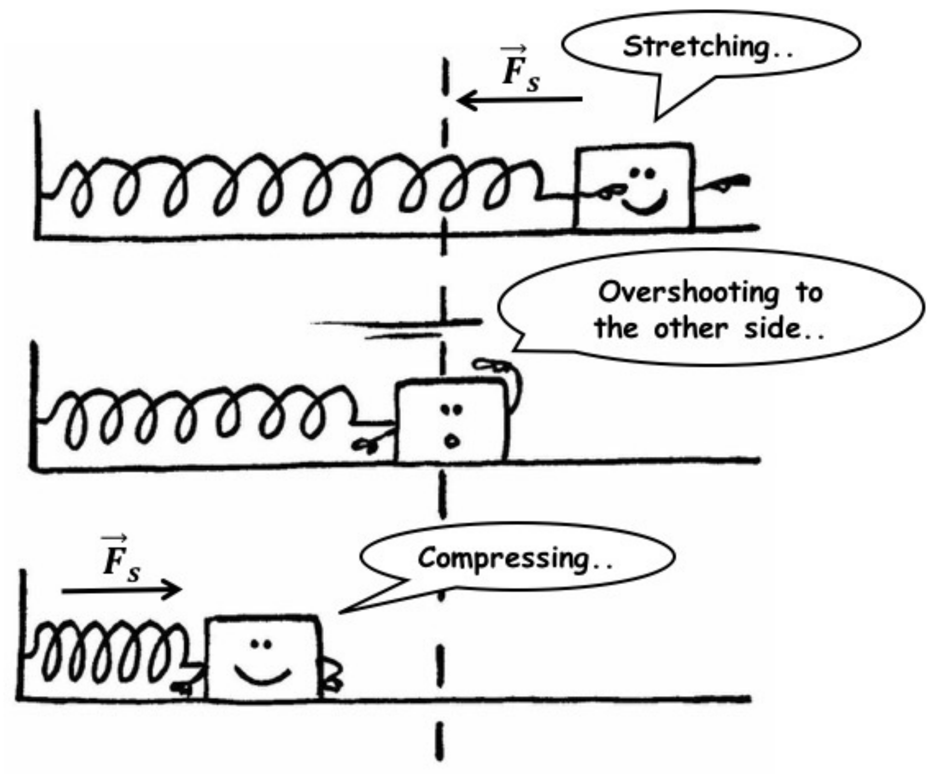
In a periodic motion, an object oscillates back and forth about an equilibrium position due to a restoring force or torque that always pulls the system back towards its equilibrium position no matter in which direction it is displaced.

A special kind of the restoring force is one that is directly proportional to the position of the object from the equilibrium position and that is always directed towards that equilibrium position. If any damping such as friction is neglected and if the restoring force is the only force acting on the system, then the motion is known as a simple harmonic motion (SHM). This kind of motion serves as a model system to analyze many oscillation problems.

Mass Attached to a Spring:

As we have seen earlier for the mass-spring system, if the block of mass m is displaced to the right or left from its equilibrium position at $x = 0$, then for small displacements, the restoring force acting on the block by the spring is given by Hook's law

$$F_s = -kx$$



If the block is displaced slightly to the left, the restoring force will accelerate it to the right transferring its PE into KE. When it reaches $x = 0$, all of its PE is converted into KE and it will overshoot to the right where the restoring force will decelerate it until all of its KE is converted into PE. At $x = -A$, it stops and then accelerates back to the left towards $x = 0$, overshooting again to $x = A$. The processes will repeat itself and the block will keep oscillating back and forth between $x = A$ and $x = -A$ if friction is neglected. By using Newton's second law, we have

$$ma = -kx$$

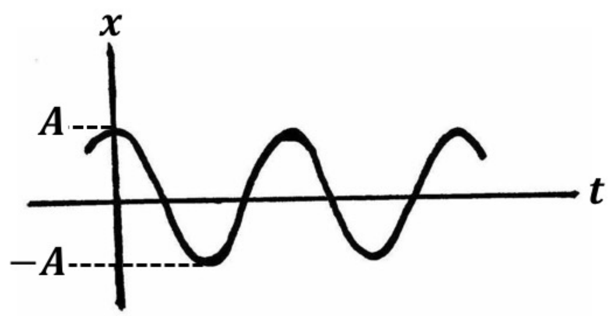
$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

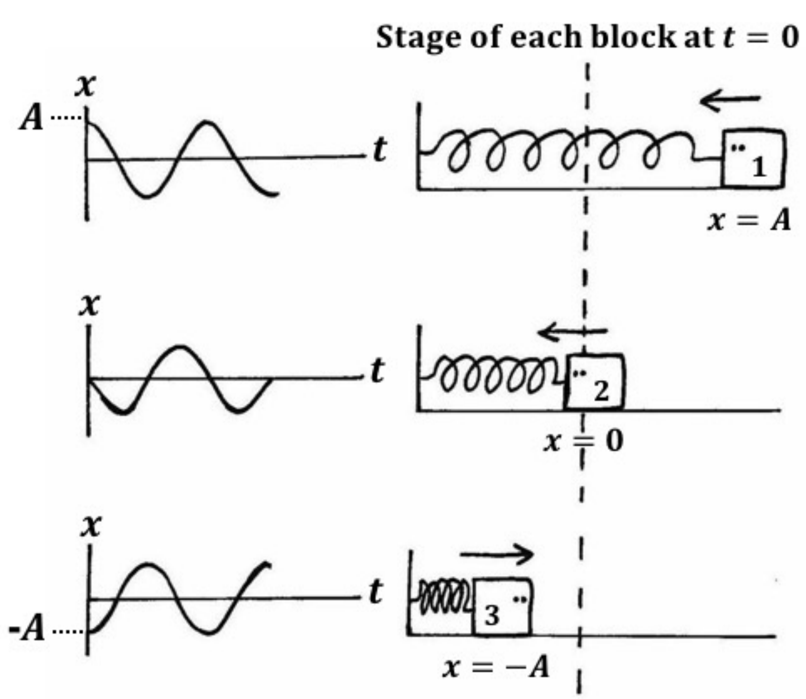
where $\omega_n = \sqrt{k/m}$ is called the natural frequency of the system and it has the unit of rad/s. A solution of this equation can be written as

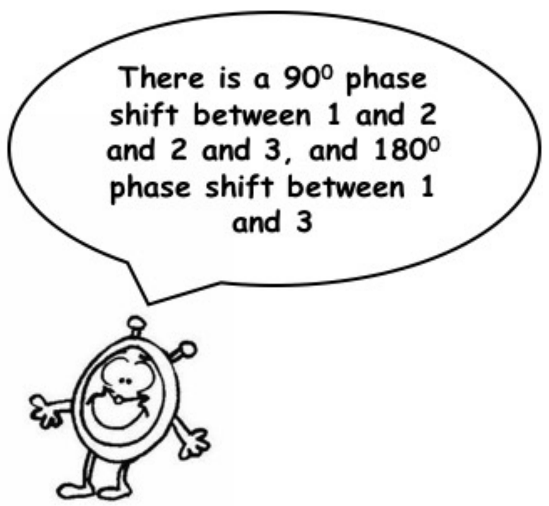
$$x(t) = A\cos(\omega_n t + \varphi)$$

where A is the amplitude (the maximum value of the position of the particle from equilibrium) and φ is the phase constant (it tells us at which stage of the cycle the motion was at $t = 0$). Both A and φ can be found from the initial conditions of the motion, i.e. from the position and velocity of the block (or any particle in SHM) at $t = 0$. For example, if at $t = 0$, the particle is at its maximum position $x = A$, then $\varphi = 0$ and the graphical representation of this motion is shown below



Suppose we have three identical oscillating mass-spring systems that have the same amplitude and frequency but differ in the phase constant. This means that at any instant of time, each block is at a different stage of its motion. When 1 is at the maximum position, 2 is at zero and 3 is at the maximum position in the opposite direction.





The period T of motion is the time required for the particle to complete one full cycle, for example from A to $-A$ and back to A. The quantity in the bracket $(\omega_n + \varphi)$ is known as the phase angle, and every time it increases by 2π after a period T, all physical quantities such as the displacement and velocity will repeat themselves. Therefore, we have

$$(\omega_n(t+T)+\varphi)-(\omega_n t+\varphi)=2\pi$$

and hence the period of motion in terms of angular frequency is

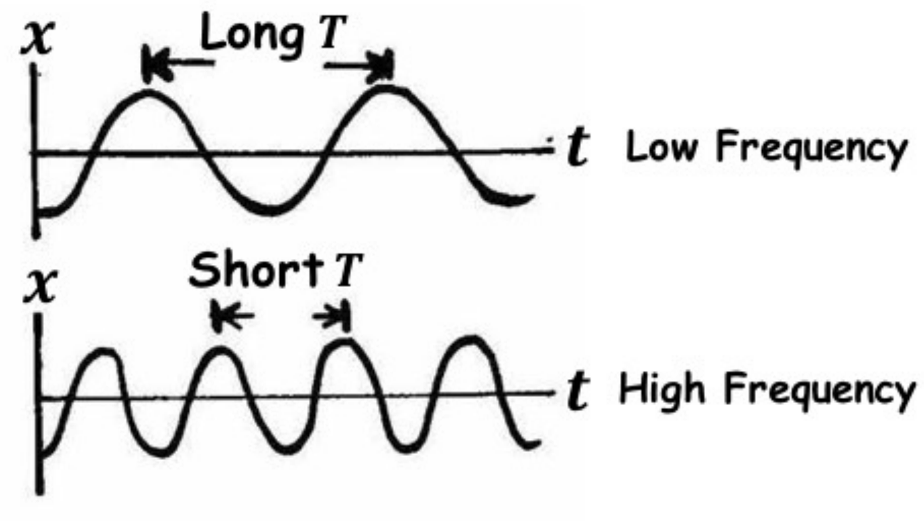
$$T=\frac{2\pi}{\omega_n}$$

The inverse of the period is known as the frequency f

$$f=\frac{1}{T}=\frac{\omega_n}{2\pi}$$

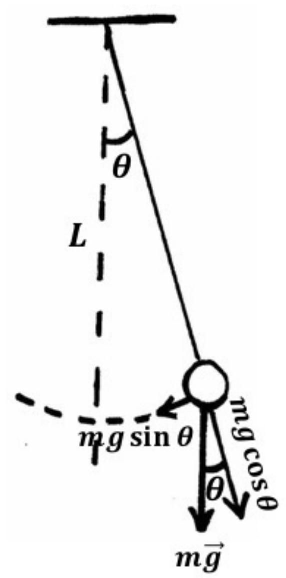
Its unit is cycle per seconds, also known as hertz (Hz). For the mass-spring system, $T=2\pi\sqrt{m/k}$, so for example if the mass is increased to $2m$, the new period is $\sqrt{2}T$.

A honeybee wings can beat at a frequency of 250 Hz (250 beats/s)



The Simple Pendulum:

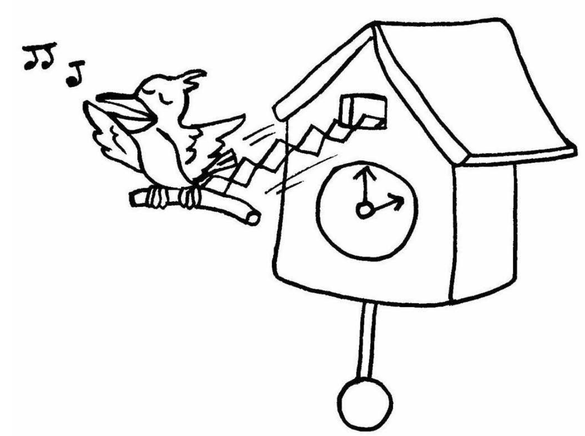
In the case of a pendulum, the component of the gravitational force that is tangent to the path ($mg\sin\theta$) is what provides the restoring force. For small angular displacements ($\leq 10^{\circ}$), the approximation $\sin\theta \approx \theta$ applies and the equation of motion may be written as



$$\frac{d^2\theta}{dt^2} + \omega_n^2\theta = 0$$

which is the form of a SHM, where $\omega_n = \sqrt{g/L}$. The period of the pendulum is $T = 2\pi\sqrt{L/g}$ and therefore a pendulum can be used to measure the gravitational acceleration g where small variations in g over an area can provide information about mineral and oil deposits, which is what Penny and Bud were trying to find!

Since the period of a pendulum is approximately independent of the amplitude for small angular displacements, it can be useful as a timekeeper in pendulum clocks, since as the amplitude decreases a bit as it runs, the clock nearly keeps the correct time.



Damped Oscillations:

The simple harmonic oscillator is an ideal system in which any damping forces such as friction or air resistance are neglected and the system will oscillate indefinitely. In real systems, there are always nonconservative dissipative forces and the oscillations diminish in time. Such motion is known as a damped oscillation and the total mechanical energy of the system is not conserved. The pendulum clock compensates for the loss in energy due to friction by using the PE in coiled-up springs or in hanging weights that lower slowly, transferring their PE into the pendulum. But eventually when the weights reach the bottom or the springs unwound, they need to be returned to their positions manually.

The simplest case is when the damping force is directly proportional to the velocity of the object relative to the medium, in a direction opposite to it, given by $\vec{R} = -b\vec{v}$, where b is a constant describing the strength of the resistive force. This kind of force is observed in a viscous fluid flow or in an object slowly oscillating in air. Applying Newton's second law we have

$$-kx - bv = ma$$

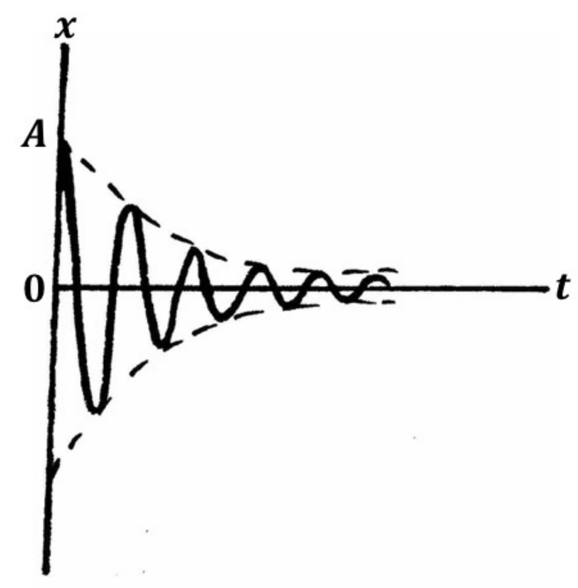
and

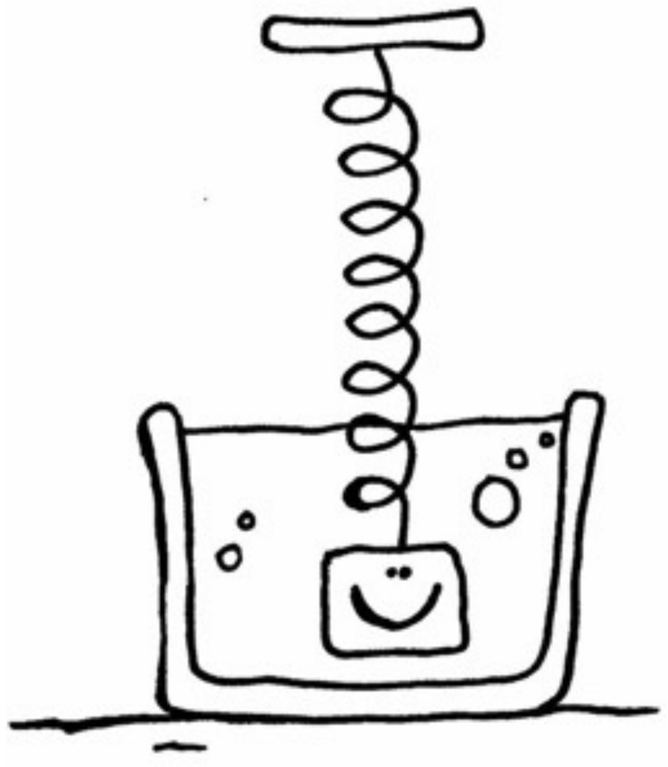
$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_n^2 x = 0$$

where $\gamma = b/m$ and has the unit of s^{-1}. If the resistive force is small such that $\gamma < 2\omega_n$, the system is lightly damped and the solution of the differential equation has the form

$$x = \left(Ae^{-\frac{\gamma}{2}t}\right)\cos(\omega_D t + \varphi)$$

where $\omega_D = \left(\omega_n^2 - \frac{\gamma^2}{4}\right)^{1/2}$ is the angular frequency of the damped oscillator and ω_n is the natural angular frequency of the oscillator in the absence of damping. The graph to the left shows the position-time graph for a damped oscillator.





An object suspended from a spring and submerged in a viscous fluid is an example of a damped oscillator where the resistive force of the fluid is of the form $\vec{R} = -b\vec{v}$

Forced Oscillations:

If an external force is applied to the damped harmonic oscillator, doing positive work on it and compensating for the loss of energy, then the resulting motion is known as a forced oscillation. A common example is when the driving force varies periodically such as $F = F_o \sin \omega t$, where ω is the angular frequency of the driving force and F_o is a constant. In general, the frequency ω of the driving force is different from the natural frequency of the oscillator ω_n if it is pushed from equilibrium and left alone to oscillate.

In the case of forced oscillation, the oscillator will oscillate at the angular frequency ω of the driving force which can be different from ω_n. Applying Newton's second law gives

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_o \sin \omega t$$

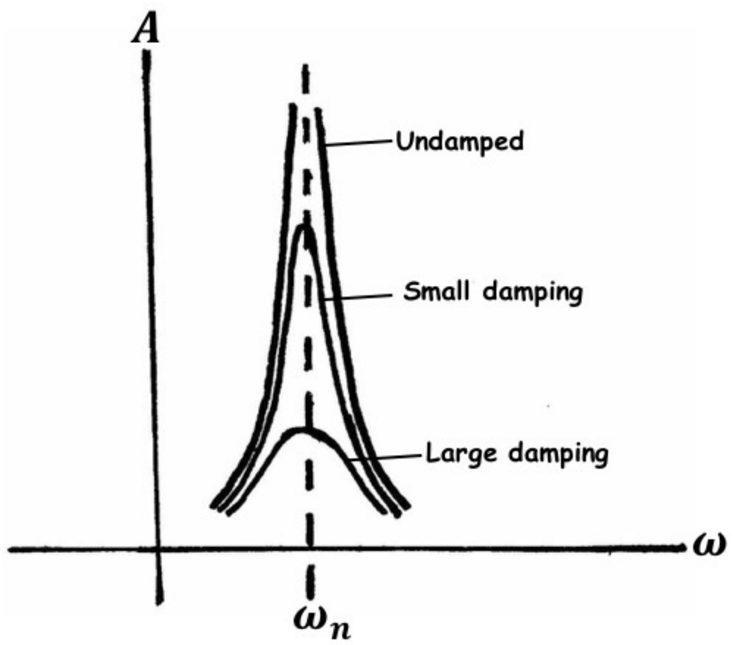
and it has the solution

$$x = A\cos(\omega t + \varphi)$$

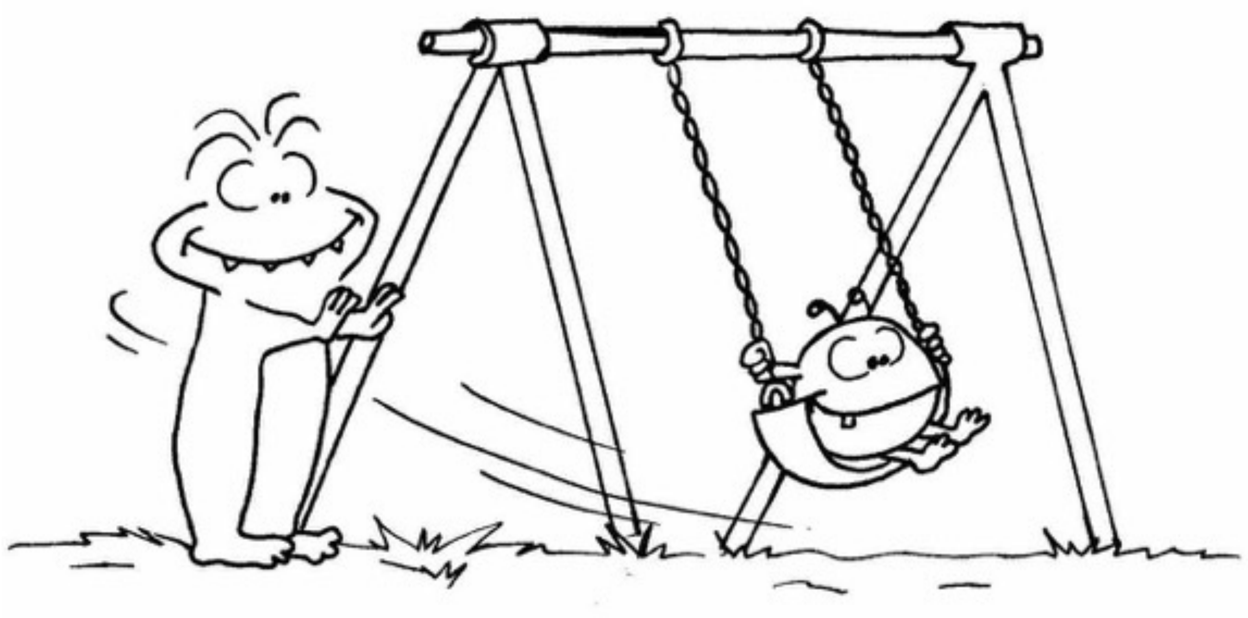
where

$$A = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_n^2) + \left(\frac{b\omega}{m}\right)^2}}$$

and $\omega_n = \sqrt{k/m}$ is the natural angular frequency of the oscillator without damping. This represents the steady state solution when the driving force is applied long enough such that the input energy from the driving force per cycle equals to the loss of mechanical energy from damping in each cycle and the object will then oscillate with a constant amplitude. If the frequency of the driving force is varied, the amplitude becomes largest when this frequency ω approaches the natural frequency of the oscillator ω_n. This phenomenon is known as resonance and the natural frequency ω_n is called the resonance frequency. The graph below shows the dramatic increase of the amplitude near the resonance frequency, where for small damping it is a sharp peak, while for heavier damping, the peak becomes broader with less height and shifts towards lower frequencies. Note that in the case of an undamped oscillator ($b = 0$), the steady state amplitude reaches infinity as $\omega \to \omega_n$. Resonance appears everywhere in physics, for example the electric circuit of a radio responds strongly to a wave having a frequency matching its resonance frequency, and this is why you may select one particular station.



Resonance can happen between a structure on the ground like a building and the shaking of the ground during an Earthquake which provides the periodic driving force. If the frequency of this force matches any of the natural frequencies of the building, it will be set into a resonance oscillation that may have a large enough amplitude to damage the building. Therefore, this problem is overcome by either designing the building to have natural frequencies outside of the range of an Earthquake or by increasing the damping within the building which will lower its response, spreading it into a larger frequency range with smaller amplitudes. Another example is the swing, where if you push your friend with a frequency that matches the natural frequency of the swing, then the amplitude of oscillation will grow.



Proof